7 İKİ KATLI İNTEGRALLER

7.2 İKİ KATLI İNTEGRALLER

TANIM Eğer

$$\lim_{\|P\|\to 0} \sum_{k=1}^{n} f\left(x_k^*, y_k^*\right)\Delta A_k$$

limiti mevcutsa, bu limite f fonksiyonunun B üzerindeki **iki katlı integrali** denir,

$$\iint_B f(x, y)dA$$

ile gösterilir. $f(x, y)$ ifadesine **integrant**, B bölgesine de **integrasyon bölgesi** denir.

TEOREM 6.1 **Birinci Fubini Teoremi**

$B = \{(x, y):\ a \le x \le b,\ c \le y \le d\}$ ve $f:\ B \to \mathbf{R}$ fonksiyonu sürekli olsun. Bu takdirde

$$\iint_B f(x, y)dxdy = \int_a^b \left(\int_c^d f(x, y)\, dy \right) dx$$
$$= \int_c^d \left(\int_a^b f(x, y)\, dx \right) dy$$

olur.

TEOREM 6.1 **İkinci Fubini Teoremi**

$\varphi,\ \ \psi : [a, b] \to \mathbf{R}$ fonksiyonları sürekli, $\forall x \in [a, b]$ için $\varphi(x) \le \psi(x)$ ve

$$B = \{(x, y):\ a \le x \le b,\quad \varphi(x) \le y \le \psi(x)\}\ \text{olsun.}$$

$f: B \to \mathbf{R}$ fonksiyonu sürekli veya parçalı sürekli ise

$$\iint_B f(x, y)\, dxdy = \int_a^b \left(\int_{\varphi(x)}^{\psi(x)} f(x, y)dy \right) dx$$

olur.

7.2 İKİ KATLI İNTEGRALLERİN UYGULAMALARI

Alan Hesabı

$$Alan(B) = \iint_B dx\,dy$$

Hacim Hesabı

$$V = \iint_B f(x, y)\,dx\,dy$$

Kütle Hesabı

$$M = \int_B \sigma(x, y)\,dx\,dy$$

Ağırlık Merkezi Hesabı

$$\overline{x} = \frac{1}{M}\iint_B x\,\sigma(x, y)\,dx\,dy \qquad \overline{y} = \iint y\,\sigma(x, y)\,dx\,dy$$

PROBLEMLER VE ÇÖZÜMLERİ

1. Aşağıdaki eşitliklerin doğruluğunu gösteriniz.

a) $\int_0^3 \int_0^2 (4-y^2)dydx = 16$ b) $\int_0^2 \int_0^1 (x^2+2y)dydx = \frac{14}{3}$ c) $\int_0^1 \int_0^1 \frac{x^2}{1+y^2}dydx = \frac{\pi}{2}$

d) $\int_0^{\pi} \int_0^{x} x\sin y\, dydx = \frac{\pi^2}{2}+2$ e) $\int_1^{\ln 8} \int_0^{\ln y} e^{x+y}dxdy = 8\ln 8 - 16 + e$ f) $\int_1^2 \int_{1/x}^{x} \frac{x^2}{y^2}dydx = \frac{9}{4}$

g) $\int_{-3}^{3} \int_{y^2-4}^{5} (x+2y)dxdy = \frac{242}{5}$ h) $\int_0^{\pi} \int_0^{\sin x} y\, dydx = \frac{\pi}{4}$ ı) $\int_1^2 \int_y^{y^2} dxdy = \frac{5}{6}$

i) $\int_0^1 \int_2^{4-2x} dydx = 1$ j) $\int_0^1 \int_{\sqrt{y}}^{1} dxdy = \frac{1}{3}$ k) $\int_0^{3/2} \int_0^{9-4x^2} 16x\, dydx = 81$

Çözüm

a) $$\int_0^3 \int_0^2 (4-y^2)dydx = \int_0^3 4y - \frac{y^3}{3}\Big|_0^2 dx = \int_0^3 \left(8-\frac{8}{3}\right)dx = \frac{16}{3}\cdot 3 = 16$$

b) $$\int_0^2 \int_0^1 (x^2+2y)dydx = \int_0^2 x^2y + y^2\Big|_0^1 dx = \int_0^2 (x^2+1)dx = \frac{x^3}{3}+x\Big|_0^2 = \frac{8}{3}+2 = \frac{14}{3}$$

c) $$\int_0^1 \int_0^1 \frac{x^2}{1+y^2}dydx = \int_0^1 x^2\arctan y\Big|_0^1 dx = \frac{\pi}{4}\int_0^1 x^2 dx = \frac{\pi}{4}\frac{x^3}{3}\Big|_0^1 = \frac{\pi}{12}$$

d) $$\int_0^{\pi} \int_0^{x} x\sin ydydx = \int_0^{\pi} x(\cos y)\Big|_0^x dx = \int_0^{\pi} (x - x\cos x)dx = \frac{\pi^2}{2} - (x\sin x)\Big|_0^{\pi} - \int_0^{\pi} (\sin x dx)$$

$$= \frac{\pi^2}{2}(-\cos x)\Big|_0^{\pi} = \frac{\pi}{2}+2$$

e) $$\int_1^{\ln 8} \int_0^{\ln y} e^{x+y}dxdy = \int_1^{\ln 8} e^y e^x\Big|_0^{\ln y} dy = \int_1^{\ln 8} (e^y y - e^y)dy = ye^y\Big|_1^{\ln 8} - \int_1^{\ln 8} e^y dy - e^y\Big|_1^{\ln 8}$$

$$= 8\ln 8 - e - 8 + e - 8 + e = 8\ln 8 - 16 + e$$

f) $2\int\limits_1^2 \int\limits_{\frac{1}{x}}^{x} \frac{x^2}{y^2}dydx = \int\limits_1^2 -\frac{x^2}{4}\Big|_{\frac{1}{x}}^{x} dx = \int\limits_1^2 (-x+x^3)dx = -\frac{x^2}{2}+\frac{x^4}{4}\Big|_1^2 = -2+\frac{16}{4}+\frac{1}{2}-\frac{1}{4}=\frac{9}{4}$

g) $\int\limits_{-3}^{3}\int\limits_{y^2-4}^{5}(x+2y)dxdy - \int\limits_{-3}^{3}\frac{x^2}{2}+2xy\Big|_{y^2-4}^{5} dy - \int\limits_{-3}^{3}\left[\frac{25}{2}+10y-\frac{(y^2-4)^2}{2}-2y(y^2-4)\right]dy$

$= \int\limits_{-3}^{3}\frac{41}{2}+18y-\frac{y^4}{2}+4y^2-2y^3 = \frac{41}{2}y+9y^2-\frac{y^5}{10}+\frac{4}{3}y^3-\frac{y^4}{4}\Big|_{-3}^{3}$

$= 123-\frac{24^3}{5}+72=\frac{552}{5} = \int\limits_{-3}^{3}\left[\frac{25}{2}+10y-\frac{(y^2-4)^2}{2}-2y(y^2-4)\right]dy$

$= \int\limits_{-3}^{3}\frac{41}{2}+18y-\frac{y^4}{2}+4y^2-2y^3 = \frac{41}{2}y+9y^2-\frac{y^5}{10}+\frac{4}{3}y^3-\frac{y^4}{4}\Big|_{-3}^{3}$

$= 123-\frac{24^3}{5}+72=\frac{552}{5}$

h) $\int\limits_0^{\pi}\int\limits_0^{\sin x} ydydx = \int\limits_0^{\pi}\frac{\sin^2 x}{2}dx = \int\limits_0^{\pi}\frac{1-\cos 2x}{4}dx = \frac{x}{4}-\frac{\sin 2x}{8}\Big|_0^{\pi} = \frac{\pi}{4}$

ı) $\int\limits_1^2\int\limits_y^{y^2} dxdy = \int\limits_1^2 y^2-ydy = \frac{y^3}{3}-\frac{y^2}{2}\Big|_1^2 = \frac{8}{4}-2-\frac{1}{3}+\frac{1}{2}=\frac{3}{2}-\frac{7}{3}=\frac{5}{6}$

i) $\int\limits_0^1\int\limits_2^{4-2x} dydx = \int\limits_0^1\int\limits_2^{4-2x} dydx = \int\limits_0^1 (2-2x)dx = 2x-x^2\Big|_0^1 = 1$

j) $\int\limits_0^1\int\limits_{\sqrt{y}}^1 dxdy = \int\limits_0^1 (1-\sqrt{y})dy = y-\frac{2}{3}y^{\frac{3}{2}}\Big|_0^1 = 1-\frac{2}{3}=\frac{1}{3}$

k) $\int\limits_0^{\frac{3}{2}}\int\limits_0^{9-4x^2} 16xdydx = \int\limits_0^{\frac{3}{2}} 16x(9-4x^2)dx = 16\left(\frac{9}{2}x^2-x^4\Big|_0^{\frac{3}{2}}\right) = 81$

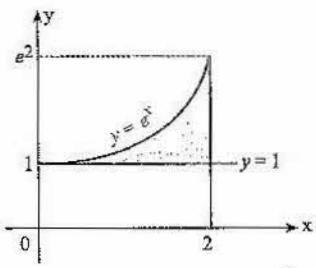
2. Aşağıdaki integrallerin
   1) İntegrasyon bölgesini çiziniz.
   2) İntegrasyon sırasını değiştiriniz.
   3) İntegrali hesaplayınız.

a) $\int_0^2 \int_1^{e^x} dydx$ b) $\int_0^1 \int_x^{2x} dydx$ c) $\int_{-2}^{1} \int_{x^2+4x}^{3x+2} dydx$

ç) $\int_0^1 \int_{\sqrt{y}}^{1} dxdy$ d) $\int_0^{\sqrt{2}} \int_{-\sqrt{4-2y^2}}^{\sqrt{4-2y^2}} ydxdy$ e) $\int_0^1 \int_{-\sqrt{y}}^{\sqrt{y}} xdxdy$

Çözüm

a) $I = \int_0^2 \int_1^{e^x} dydx$



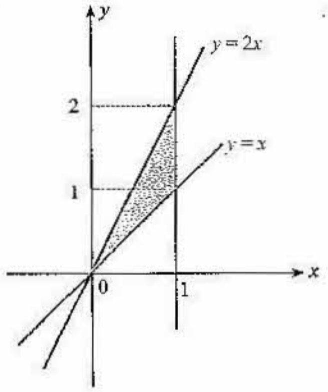
1. $y = e^x \quad x = 0$
   $y = 1 \quad x = 2$

2. $I = \int_1^{e^2} \int_{\ln y}^{2} dxdy$

3. $I = \int_0^2 (e^x - 1)dx = e^x - x \Big|_0^2 = e^2 - 2 - 1 = e^2 - 3$

b) $I = \int_0^1 \int_x^{2x} dydx$

1.

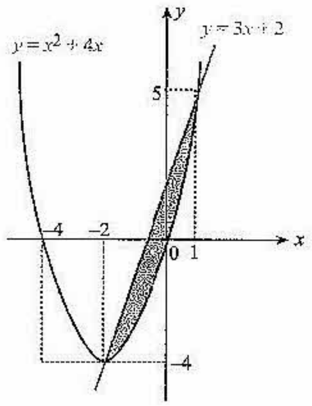


2. $I = \int_0^1 \int_{\frac{y}{2}}^{y} dxdy + \int_1^2 \int_{\frac{y}{2}}^{1} dxdy$

3. $I = \int_0^1 \int_x^{2x} dydx = \int_0^1 y\Big|_x^{2x} dx = \int_0^1 xdx = \frac{x^2}{2}\Big|_0^1 = \frac{1}{2}$

c) $\mathrm{I} = \int_{-2}^{1} \int_{x^2+4x}^{3x+2} dydx$

1.

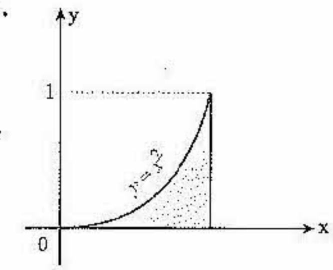


2. $\int_{-4}^{5} \int_{\frac{y-2}{3}}^{-2+\sqrt{y+4}} dxdy$

3. $\mathrm{I} = \int_{-2}^{1} (-x^2 - x + 2)dx = -\frac{x^3}{3} - \frac{x^2}{2} + 2x \Big|_{-2}^{1} = \frac{9}{2}$

ç) $\mathrm{I} = \int_{0}^{1} \int_{\sqrt{y}}^{1} dxdy$

1.

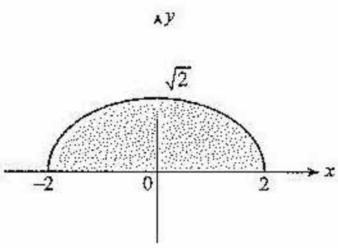


2. $\int_{0}^{1} \int_{y=0}^{x^2} dydx$

3. $\int_{0}^{1} \int_{0}^{x^2} dydx = \int_{0}^{1} x^2 dx = \frac{1}{3}$

d) $\mathrm{I} = \int_{0}^{\sqrt{2}} \int_{-\sqrt{4-2y^2}}^{\sqrt{4-2y^2}} ydxdy$

1. $x = \sqrt{4 - 2y^2} \Rightarrow x^2 + 2y^2 = 4$ elips

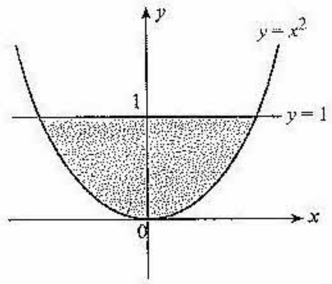


2. $\mathrm{I} = \int_{-2}^{2} \int_{0}^{\sqrt{\frac{4-x^2}{2}}} ydydx$

3. $I = \int_{-2}^{2} \int_{0}^{\sqrt{\frac{4-x^2}{2}}} ydydx = \int_{-2}^{2} \frac{4-x^2}{4} dx = x - \frac{x^3}{12} \Big|_{-2}^{2} = \frac{8}{3}$

e) $I = \int\limits_{0}^{1} \int\limits_{-\sqrt{y}}^{\sqrt{y}} x\,dx\,dy$

1.



2. $\int\limits_{-1}^{1} \int\limits_{x^2}^{1} x\,dy\,dx$

3. $I = \int\limits_{-1}^{1} \int\limits_{x^2}^{1} x\,dy\,dx = \int\limits_{-1}^{1} (x - x^3)\,dx = \frac{x^2}{2} - \frac{x^4}{4}\Big|_{-1}^{1} = 0$

3. Aşağıdaki integrallerin

a) İntegrasyon bölgesini çiziniz. b) İntegrasyon sırasını değiştiriniz.

a) $\int\limits_{0}^{1} \int\limits_{y}^{2y} f(x,y)\,dx\,dy$ b) $\int\limits_{0}^{1} \int\limits_{x^2}^{x} f(x,y)\,dy\,dx$ c) $\int\limits_{0}^{1} \int\limits_{-\sqrt{1-y^2}}^{1-y} f(x,y)\,dx\,dy$

d) $\int\limits_{0}^{2} \int\limits_{0}^{x^3} f(x,y)\,dy\,dx$ e) $\int\limits_{0}^{2} \int\limits_{\sqrt{2x-x^2}}^{2} f(x,y)\,dy\,dx$ f) $\int\limits_{0}^{1} \int\limits_{\frac{1-x^2}{2}}^{\sqrt{1-x^2}} f(x,y)\,dy\,dx$

g) $\int\limits_{1}^{2} \int\limits_{0}^{\sqrt{4x-x^2}} f(x,y)\,dy\,dx$ h) $\int\limits_{0}^{1} \int\limits_{\frac{y^2}{2}}^{\sqrt{3-y^2}} f(x,y)\,dx\,dy$ ı) $\int\limits_{-10}^{0} \int\limits_{0}^{1+\sqrt{100-x^2}} f(x,y)\,dy\,dx$

i) $\int\limits_{0}^{a\sqrt{2}/2} \int\limits_{0}^{x} f(x,y)\,dy\,dx + \int\limits_{\frac{a\sqrt{2}}{2}}^{a} \int\limits_{0}^{\sqrt{a^2-x^2}} f(x,y)\,dy\,dx$

j) $\int\limits_{0}^{4} \int\limits_{\sqrt{y}}^{\sqrt{2y}} f(x,y)\,dx\,dy + \int\limits_{4}^{8} \int\limits_{\frac{y}{2}}^{\sqrt{2y}} f(x,y)\,dx\,dy$

k) $\int\limits_{0}^{2} \int\limits_{x}^{2x} f(x,y)\,dy\,dx + \int\limits_{2}^{3} \int\limits_{x}^{6-x} f(x,y)\,dy\,dx$

l) $\int\limits_{0}^{1} \int\limits_{y}^{1} f(x,y)\,dx\,dy + \int\limits_{-1}^{0} \int\limits_{-y}^{1} f(x,y)\,dx\,dy$

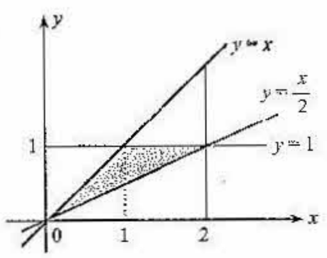
m) $\int\limits_{0}^{1} \int\limits_{\frac{3}{4}x^2}^{4x^2} f(x,y)\,dy\,dx + \int\limits_{1}^{2} \int\limits_{\frac{3}{4}x^2}^{5-x} f(x,y)\,dy\,dx$

n) $\int\limits_{0}^{1} \int\limits_{-\sqrt{y}}^{\sqrt{y}} f(x,y)\,dx\,dy + \int\limits_{1}^{4} \int\limits_{-\sqrt{y}}^{2-y} f(x,y)\,dx\,dy$

o) $\int\limits_{0}^{1} \int\limits_{\frac{y^2}{4}}^{y^2} f(x,y)\,dx\,dy + \int\limits_{1}^{4} \int\limits_{\frac{y^2}{4}}^{y} f(x,y)\,dx\,dy$

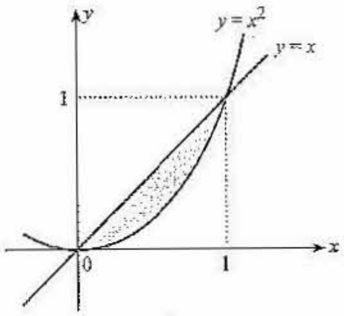
Çözüm

a) 1.



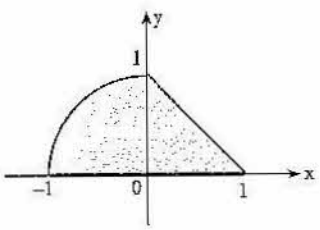
2. $\displaystyle\int_0^1\int_{\frac{x}{2}}^{x} f(x,y)dydx + \int_1^2\int_{\frac{x}{2}}^{1} f(x,y)dydx$

b) 1.



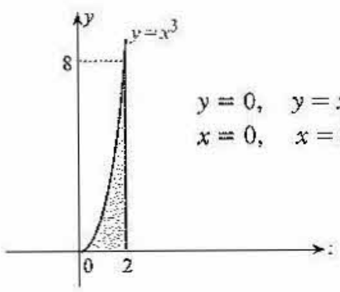
2. $\displaystyle\int_0^1\int_y^{\sqrt{y}} f(x,y)dxdy$

c) 1.



2. $\displaystyle I = \int_{-1}^{0}\int_{y=0}^{\sqrt{1-x^2}} f(x,y)dydx + \int_0^1\int_0^{1-x} f(x,y)dydx$

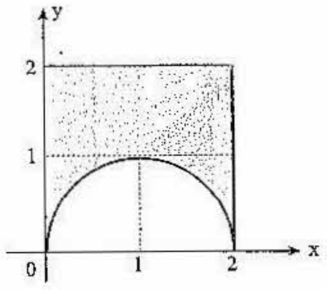
d) 1.



$y = 0, \quad y = x^3$
$x = 0, \quad x = 2$

2. $\displaystyle\int_0^8\int_{\sqrt[3]{y}}^{2} f(x,y)dxdx$

e) **1.**

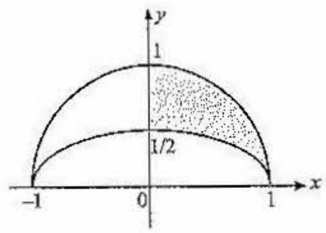


$$y = \sqrt{2x - x^2} \Rightarrow y^2 = 2x - x^2 \Rightarrow$$
$$x^2 - 2x + y^2 = 0 \Rightarrow (x-1)^2 + y^2 = 1$$

2. $$I = \int_0^1 \int_0^{1-\sqrt{1-y^2}} f(x,y)dxdy + \int_1^2 \int_0^2 f(x,y)dxdy + \int_0^1 \int_{1+\sqrt{1-y^2}}^2 f(x,y)dxdy$$

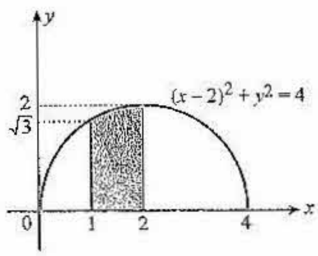
f) **1.** $y = \dfrac{1-x^2}{2}$ parabol

$y = \sqrt{1-x^2}$ üst yarı çember



2. $$\int_0^{\frac{1}{2}} \int_{\sqrt{1-2y}}^{\sqrt{1-y^2}} f(x,y)dxdy + \int_{\frac{1}{2}}^{1} \int_0^{\sqrt{1-y^2}} f(x,y)dxdy$$

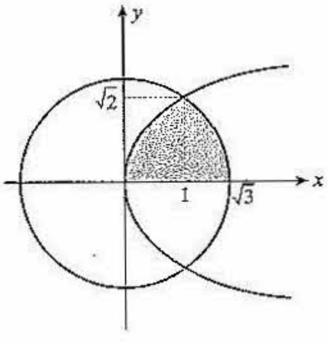
g) **1.**



$$y = \sqrt{4x - x^2} \Rightarrow y^2 = 4x - x^2 \Rightarrow$$
$$x^2 - 4x + y^2 = 0 \Rightarrow (x-2)^2 + y^2 = 4$$

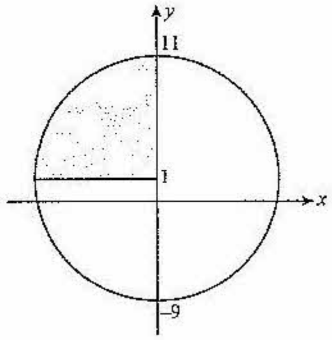
2. $$I = \int_0^{\sqrt{3}} \int_1^2 f(x,y)dxdy + \int_{\sqrt{3}}^{2} \int_{2-\sqrt{4-y^2}}^{2} f(x,y)dxdy$$

h) **1.**



2. $$\int_0^1 \int_0^{\sqrt{2x}} f(x,y)dydx + \int_1^{\sqrt{3}} \int_0^{\sqrt{3-x^2}} f(x,y)dydx$$

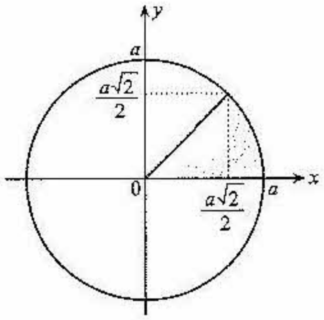
ı) **1.**



$$x = -\sqrt{99-2y-y^2} \Rightarrow$$
$$x^2 + (y-1)^2 = 100$$

2. $I = \int\limits_{1}^{11} \int\limits_{-\sqrt{99-2y-y^2}}^{0} f(x,y)\,dxdy$

i) **1.**



2. $I = \int\limits_{0}^{\frac{a\sqrt{2}}{2}} \int\limits_{y}^{\sqrt{a^2-x^2}} f(x,y)\,dxdy$

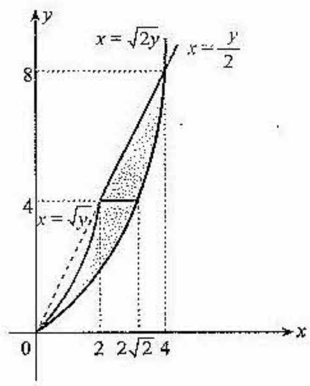
j) **1.** $x = \sqrt{y} \Rightarrow y = x^2$

$x = \sqrt{2y} \Rightarrow y = \frac{1}{2}x^2$

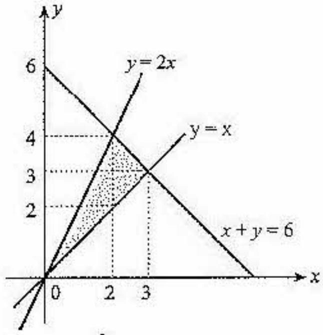
$y = 0,\ y = 4$

$x = \frac{y}{2} \Rightarrow y = 2x$

2. $\int\limits_{0}^{2} \int\limits_{\frac{x^2}{2}}^{x^2} f(x,y)\,dydx + \int\limits_{2}^{4} \int\limits_{\frac{x^2}{2}}^{2x} f(x,y)\,dydx$

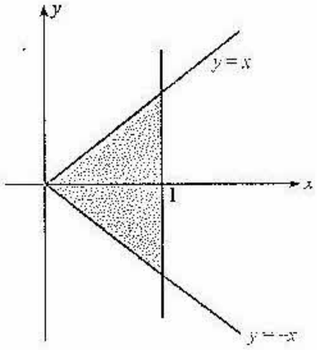


k) **1.**



2. $I = \int\limits_{0}^{3} \int\limits_{x=\frac{y}{2}}^{y} f(x,y)\,dxdy + \int\limits_{3}^{4} \int\limits_{\frac{y}{2}}^{6-y} f(x,y)\,dxdy$

l) 1.



2. $\displaystyle\int_0^1 \int_{-x}^{x} f(x,y)\,dy\,dx$

4. $\displaystyle\int_0^1 \int_0^1 \frac{y-x}{(x+y)^3}\,dx\,dy$ ve $\displaystyle\int_0^1 \int_0^1 \frac{y-x}{(x+y)^3}\,dy\,dx$

integrallerinin farklı değerlere sahip olduğunu gösteriniz. Bu sonuç Fubini Teoremiyle çelişir mi?

Çözüm

$$\int_0^1 \int_0^1 \frac{y-x}{(x+y)^3}dxdy = \frac{1}{2}, \qquad \int_0^1 \int_0^1 \frac{y-x}{(x+y)^3}dydx = -\frac{1}{2}$$

olduğu gösterilebilir. Eğer Mathematica, Maple veya Derive gibi bir program kullanılırsa bu sonuçlar daha hızlı elde edilebilir.

Bu sonuç Fubini Teoremiyle çelişmez, zira $f(x,y) = \dfrac{y-x}{(x+y)^3}$ fonksiyonu $[0, 1] \times [0, 1]$ bölgesi üzerinde sürekli değildir.

5. Aşağıdaki integralleri hesaplayınız.

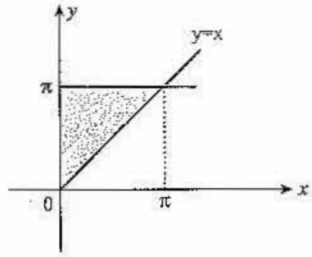
a) $\displaystyle\int_0^{\pi} \int_x^{\pi} \frac{\sin y}{y}\,dydx$ b) $\displaystyle\int_0^1 \int_{2y}^{2} \cos(x^2)\,dxdy$ c) $\displaystyle\int_0^1 \int_y^1 x^2 e^{xy}\,dxdy$ d) $\displaystyle\int_0^2 \int_{y/2}^{1} e^{x^2}\,dxdy$

e) $\displaystyle\int_0^2 \int_x^2 2y^2 \sin(xy)\,dydx$ f) $\displaystyle\int_0^8 \int_{\sqrt[3]{x}}^{2} \frac{dydx}{y^4+1}$ g) $\displaystyle\int_0^2 \int_0^{4-x^2} \frac{xe^{2y}}{4-y}\,dydx$ h) $\displaystyle\int_0^{2\sqrt{\ln 3}} \int_{y/2}^{\sqrt{\ln 3}} e^{x^2}\,dxdy$

ı) $\displaystyle\int_0^3 \int_{\sqrt{\frac{x}{3}}}^{1} e^{y^3}\,dydx$ i) $\displaystyle\int_{-1}^{0} \int_{-y}^{1} e^{x^2}\,dxdy + \int_0^1 \int_y^1 e^{x^2}\,dxdy$

Çözüm

a) İntegrasyon bölgesi yanda turuncu renkte gösterilmiştir.

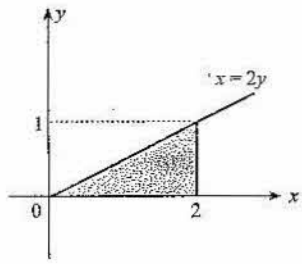


$$I=\int_0^{\pi}\int_x^{\pi}\frac{\sin y}{y}dydx$$

$$=\int_0^{\pi}\int_0^{y}\frac{\sin y}{y}dxdy=\int_0^{\pi}\frac{\sin y}{y}ydy$$

$$=\int_0^{\pi}\sin ydy=-\cos y\Big|_0^{\pi}=2$$

b) İntegrasyon bölgesi yandaki yeşil bölgedir.

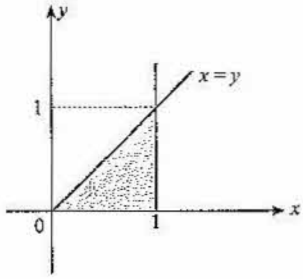


$$I=\int_0^{1}\int_{2y}^{2}\cos(x^2)dxdy$$

$$=\int_0^{2}\int_0^{\frac{x}{2}}\cos(x^2)dydx=\int_0^{2}\frac{x}{2}\cos(x^2)dx$$

$$=\frac{1}{4}\sin(x^2)\Big|_0^{2}=\frac{\sin 4}{4}$$

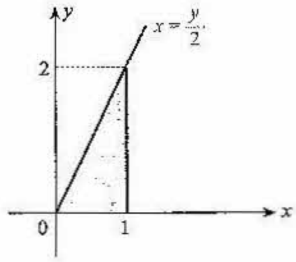
c) İntegrasyon bölgesi yandaki pembe bölgedir.



$$I=\int_0^{1}\int_y^{1}x^2e^{xy}dxdy=\int_0^{1}\int_0^{x}x^2e^{xy}dydx=\int_0^{1}xe^{xy}\Big|_0^{x}dx$$

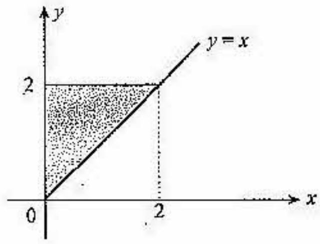
$$=\int_0^{1}(xe^{x^2}-x)dx=\left(\frac{1}{2}e^{x^2}-\frac{x^2}{2}\right)\Big|_0^{1}=\frac{1}{2}(e-1)-\frac{1}{2}=\frac{e}{2}-1$$

d) $I=\int_0^{2}\int_{\frac{y}{2}}^{1}e^{x^2}dxdy$



$$I=\int_0^{1}\int_0^{2x}e^{x^2}dydx=\int_0^{1}2xe^{x^2}dx=e^{x^2}\Big|_0^{1}=e-1$$

e) $I = \int\limits_0^2 \int\limits_x^2 2y^2 \sin(xy)\,dydx$

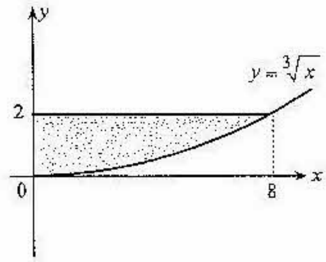


$$I = \int\limits_0^2 \int\limits_0^y 2y^2 \sin(xy)\,dxdy$$

$$= \int\limits_0^2 2y\left(-\cos(xy)\Big|_0^y\right)dy = \int\limits_0^2 \left(-2y\cos(y^2) + 2y\right)dy$$

$$= -\sin(y^2) + y^2\Big|_0^2 = 4 - \sin 4$$

f) $I = \int\limits_0^8 \int\limits_{\sqrt[3]{x}}^2 \dfrac{dydx}{y^4+1}$

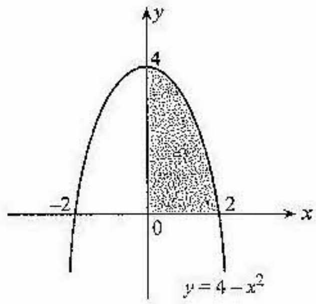


$$I = \int\limits_0^2 \int\limits_0^{y^3} \frac{dxdy}{y^4+1} = \int\limits_0^2 \frac{y^3}{1+y^4}dy$$

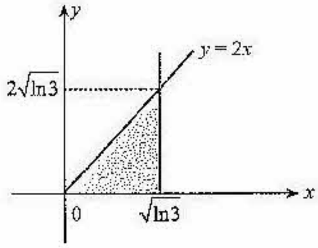
$$= \frac{1}{4}\ln(1+y^4)\Big|_0^2 = \frac{\ln 17}{4}$$

g) $I = \int\limits_0^2 \int\limits_0^{4-x^2} \dfrac{xe^{2y}}{4-y}dydx = \int\limits_0^4 \int\limits_0^{\sqrt{4-y}} \dfrac{xe^{2y}}{4-y}dxdy$

$$= \frac{1}{2}\int\limits_0^4 \frac{e^{2y}}{4-y}(4-y)dy = \frac{1}{2}\int\limits_0^4 e^{2ydy} = \frac{e^8-1}{4}$$



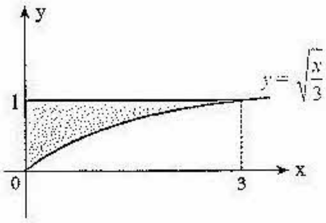
h) $I = \int\limits_0^{2\sqrt{\ln 3}} \int\limits_{\frac{y}{2}}^{\sqrt{\ln 3}} e^{x^2}dxdy = \int\limits_0^{\sqrt{\ln 3}} \int\limits_0^{2x} e^{x^2}dydx$



$$= \int\limits_0^{\sqrt{\ln 3}} e^{x^2}2xdx = e^{x^2}\Big|_0^{\sqrt{\ln 3}} = 3 - 1 = 2$$

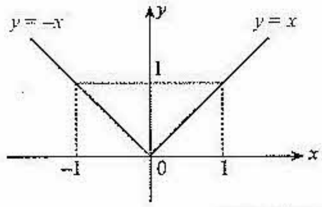
h) $I=\int\limits_{0}^{3}\int\limits_{\sqrt{\frac{x}{3}}}^{1} e^{y^3}dydx=\int\limits_{0}^{1}\int\limits_{0}^{3y^2} e^{y^3}dxdy=\int\limits_{0}^{1} e^{y^3}3y^2dy$

$=e^{y^3}\Big|_0^1=e-1$



i) $\int\limits_{-1}^{0}\int\limits_{-y}^{1} e^{x^2}dxdy+\int\limits_{0}^{1}\int\limits_{y}^{1} e^{x^2}dxdy=\int\limits_{0}^{1}\int\limits_{-x}^{x} e^{x^2}dydx=\int\limits_{0}^{1} 2xe^{x^2}dx$

$=e^{x^2}\Big|_0^1=e-1$



6. Aşağıdaki integralleri hesaplayınız.

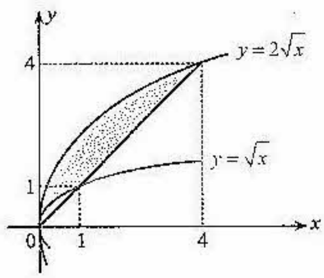
a) $\int\limits_{0}^{1}\int\limits_{\frac{y^2}{4}}^{y^2} dxdy+\int\limits_{1}^{4}\int\limits_{\frac{y^2}{4}}^{y} dxdy$ b) $\int\limits_{0}^{\sqrt{2}/2}\int\limits_{0}^{x} dydx+\int\limits_{\sqrt{2}/2}^{1}\int\limits_{0}^{\sqrt{1-x^2}} dydx$

Çözüm

a) $I=\int\limits_{0}^{1}\int\limits_{\frac{y^2}{4}}^{y^2} dxdy+\int\limits_{1}^{4}\int\limits_{\frac{y^2}{4}}^{y} dxdy=\int\limits_{0}^{1}\int\limits_{\sqrt{x}}^{2\sqrt{x}} dydx+\int\limits_{1}^{4}\int\limits_{x}^{2\sqrt{x}} dydx$

$=\int\limits_{0}^{1}\sqrt{x}\,dx+\int\limits_{1}^{4}(2\sqrt{x}-x)dx=\frac{2}{3}x^{\frac{3}{2}}\Big|_0^1+\left(\frac{4}{3}x^{\frac{3}{2}}-\frac{x^2}{2}\right)\Big|_1^4$

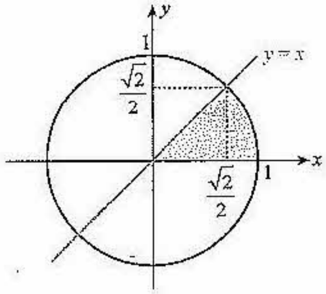
$=\frac{2}{3}+\frac{11}{6}=\frac{5}{2}$



b) $I=\int\limits_{0}^{\frac{\sqrt{2}}{2}}\int\limits_{y}^{\sqrt{1-y^2}} xdxdy=\int\limits_{0}^{\frac{\pi}{4}}\int\limits_{0}^{1} r\cos\theta rdrd\varphi$

$=\int\limits_{0}^{\frac{\pi}{4}}\frac{r^3}{3}\Big|_0^1\cos\theta d\theta=\frac{1}{3}\int\limits_{0}^{\frac{\pi}{4}}\cos\theta d\theta$

$=\frac{1}{3}\sin\theta\Big|_0^{\frac{\pi}{4}}=\frac{1}{3}\frac{\sqrt{2}}{2}=\frac{\sqrt{2}}{6}$ *olur.*



PROBLEMLER VE ÇÖZÜMLERİ

1. B integrasyon bölgesi, birinci bölgede $x^2-y^2=1$, $x^2-y^2=9$, $xy=2$ ve $xy=4$ hiperbolleri tarafından sınırlanan bölge olduğuna göre,

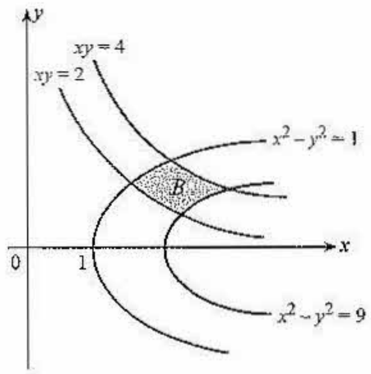
$$\iint_B (x^2+y^2)dxdy$$

integralini hesaplayınız.

Çözüm

$$u = xy$$

$$v = x^2-y^2$$



dönüşümünü yapalım.

$xy=2$ ve $xy=4$ hiperbolleri $u=2$ ve $u=4$ doğrularına, $x^2-y^2=1$ ve $x^2-y^2=9$ hiperbolleri de $v=1$ ve $v=9$ doğrularına dönüşür.

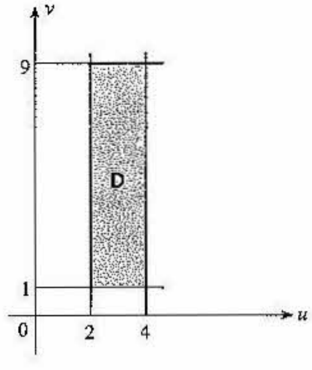
Bu durumda B bölgesi de

$D=\{(u,v):2\le u\le 4,\ 1\le v\le 9\}$ bölgesine dönüşür.

$$J=\frac{\partial(x,y)}{\partial(u,v)}=\frac{1}{\begin{vmatrix} y & x \\ 2x & -2y\end{vmatrix}}=\frac{-1}{2(x^2+y^2)}$$

olduğundan

$$\iint_B (x^2+y^2)dxdy=\iint_D x^2+y^2|J|dudv$$



$$=\int_{v=1}^{9}\int_{u=2}^{4}(x^2+y^2)\frac{1}{2(x^2+y^2)}dudv$$

$$=\frac{1}{2}\int_1^9\int_2^4 dudv=\frac{1}{2}(4-2)(9-1)$$

$$=8$$

bulunur.

2. B integrasyon bölgesi $x^2+y^2 \leq 1$ dairesi olduğuna göre,

$$\iint_B (1-x^2-y^2)\,dxdy$$

integralini kutupsal koordinatlara geçerek hesaplayınız.

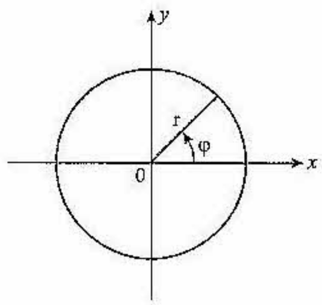
Çözüm

Kutupsal koordinatlara geçilerek

$$\iint_B (1-x^2-y^2)dxdy = \int_{\theta=0}^{2\pi}\int_{r=0}^{1}(1-r^2)rdrd\theta = \int_0^{2\pi}\left.\frac{r^2}{2}-\frac{r^4}{4}\right|_0^1 d\theta$$

$$= \frac{1}{4}\int_0^{2\pi} d\theta = \frac{\pi}{2}$$

bulunur.



3. $B=\{(x,y):\ 1\leq x\leq 2 \quad \text{ve} \quad 0\leq y\leq x\}$ olduğuna göre,

$$\iint_B \frac{y\sqrt{x^2+y^2}}{x}\,dx\,dy$$

integralini kutupsal koordinatlar yardımıyla hesaplayınız.

Çözüm

Kutupsal koordinatlara geçilirse

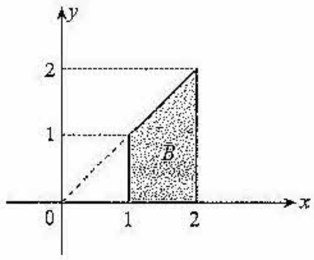
$$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases}$$

$$x = 2 \Rightarrow 2 = r\cos\theta \Rightarrow r = \frac{2}{\cos\theta}$$

$$x = 1 \Rightarrow 1 = r\cos\theta \Rightarrow r = \frac{1}{\cos\theta}$$

$$y = x \Rightarrow r\cos\theta = r\sin\theta \Rightarrow \theta = \frac{\pi}{4}$$

bulunur.



$$\iint_B \frac{y\sqrt{x^2+y^2}}{x}dxdy = \int_{\theta=0}^{\frac{\pi}{4}} \int_{r=\frac{1}{\cos\theta}}^{\frac{2}{\cos\theta}} \frac{r\sin\theta\sqrt{r^2}}{ros\theta} rdrd\theta = \int_{\theta=0}^{\frac{\pi}{4}} \int_{r=\frac{1}{\cos\theta}}^{\frac{2}{\cos\theta}} r^2 \tan\theta drd\theta$$

$$= \frac{1}{3}\int_{\theta=0}^{\frac{\pi}{4}} \frac{8\sin\theta}{\cos^4\theta} - \frac{\sin\theta}{\cos^4\theta}d\theta = \frac{1}{3}\left(\frac{8}{3}\cdot\frac{1}{\cos^3\theta} - \frac{1}{3\cos^3\theta}\right)\Big|_0^{\frac{\pi}{4}}$$

$$= \frac{14\sqrt{2}}{9}.$$

4. B bölgesi, köşeleri $(\pi, 0)$, $(2\pi, \pi)$, $(\pi, 2\pi)$, $(0, \pi)$ olan paralelkenar olduğuna göre,

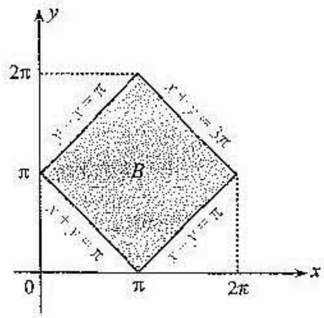
$$\iint_B (x-y)^2 \sin^2(x+y)\, dxdy$$

integralini

$$\begin{cases} u = x - y \\ v = x + y \end{cases}$$

dönüşümü yardımıyla hesaplayınız.

Çözüm



$(0, \pi)$ ve $(\pi, 2\pi)$ noktalarından geçen doğrunun denklemi $x - y = -\pi$,

$(\pi, 2\pi)$ ve $(2\pi, \pi)$ noktalarından geçen doğrunun denklemi $x + y = 3\pi$,

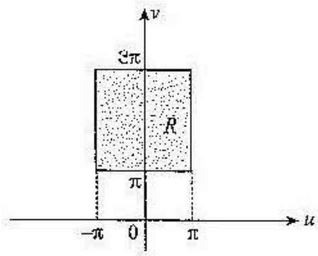
$(2\pi, \pi)$ ve $(\pi, 0)$ noktalarından geçen doğrunun denklem $x - y = \pi$,

$(\pi, 0)$ ile $(0, \pi)$ noktalarından geçen doğrunun denklem $x + y = \pi$ dir.

Bu doğrular, verilen bölge dönüşümü yardımıyla

$$u = -\pi,\ u = \pi,\ v = 3\pi \text{ ve } v = \pi$$

doğrularına dönüşür.



$$J = \frac{\partial(x,y)}{\partial(u,v)} = \frac{1}{\frac{\partial(u,v)}{\partial(x,y)}} = \frac{1}{\begin{vmatrix} 1 & -1 \\ 1 & 1 \end{vmatrix}} = \frac{1}{2}$$

olduğundan

$$\iint_B (x-y)^2\sin^2(x+y)\,dxdy = \iint_D u^2\sin^2 v|J|\,dudv = \frac{1}{2}\int_{\pi}^{3\pi}\int_{-\pi}^{\pi} u^2\sin^2 v\,dudv$$

$$= \frac{\pi^3}{3}\int_{\pi}^{3\pi}\sin^2 v\,dv = \frac{\pi^3}{3}\int_{\pi}^{3\pi}\frac{1-\cos 2v}{2}dv = \frac{\pi^3}{6}\left(v-\frac{\sin 2v}{2}\right)\Big|_{\pi}^{3\pi} = \frac{\pi^4}{3}$$

bulunur.

5. B bölgesi, birinci bölgede $x+y=1$, $x+y=2$, doğrularıyla koordinat eksenleri arasında kalan bölge olduğuna göre,

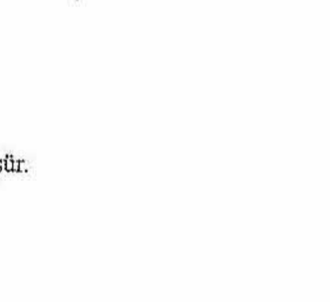
$$\iint_B \frac{(x-y)^2}{1+x+y}dxdy$$

integralini

$$\begin{cases} u = x+y \\ v = x-y \end{cases}$$

dönüşümü yardımıyla hesaplayınız.

Çözüm



$[AB]$ nin denklemi $y=0$, $1\leq x\leq 2$ dir.

Bu durumda

$$\left.\begin{array}{l} u = x+y \Rightarrow u = x \\ v = x-y \Rightarrow v = x \end{array}\right\} \Rightarrow u = v, 1 \leq u \leq 2 \ \ olur.$$

$[BC]$ nin denklemi $x+y=2$, $0\leq x\leq 2$ ve $0\leq y\leq 2$ dir.

$$u = x+y \Rightarrow u = 2 \text{ olur.}$$

$$v = x-y \Rightarrow -2\leq x-y\leq 2 \Rightarrow -2\leq v\leq 2 \text{ olur.}$$

$[CD]$ nin denklemi $x=0$, $1\leq y\leq 2$ olur. Bu durumda $[CD]$ doğru parçası

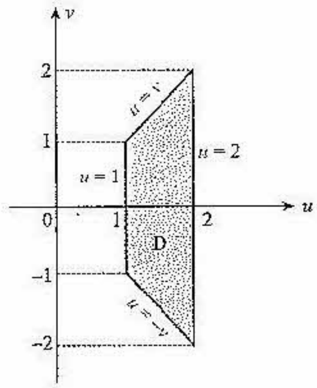
$$\left.\begin{array}{l} u = x+y \Rightarrow u = y \\ v = x-y \Rightarrow v = -y \end{array}\right\} v = -u, \ \ 1 \leq u \leq 2 \ \text{ doğru parçasına dönüşür.}$$

$[DA]$ nın denklemi $y+x=1 \quad 0 \le x \le 1 \quad 0 \le y \le 1$ dir.

Bu doğru parçası

$$u = x + y \Rightarrow u = 1$$
$$v = x - y \Rightarrow -1 \le x - y \le 1 \Rightarrow -1 \le v \le 1$$

doğru parçasına dönüşür.



$$J = \frac{1}{\frac{d(u,v)}{d(x,y)}} = \frac{1}{\begin{vmatrix} 1 & 1 \\ 1 & -1 \end{vmatrix}} = -\frac{1}{2} \Rightarrow |J| = \frac{1}{2} \text{ olduğundan}$$

$$I = \int_1^2 \int_{-u}^{u} \frac{v^2}{1+u}\frac{1}{2} dv du = \frac{1}{2}\int_1^2 \frac{1}{1+u}\frac{v^3}{3}\Bigg|_{-u}^{u} du = \frac{1}{3}\int_1^2 \frac{u^3}{1+u} du$$

$$= \frac{1}{3}\left(\frac{u^3}{3} - \frac{u^2}{2} + u - \ln(1+u)\Bigg|_1^2\right) = \frac{1}{3}\left(\frac{11}{6} - \ln\frac{3}{2}\right)$$

bulunur.

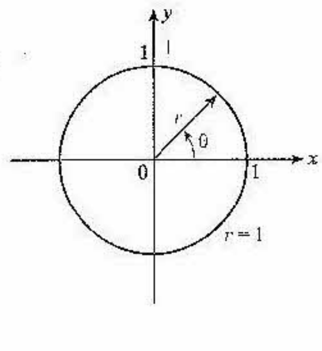
6. Kutupsal koordinatlardan yararlanarak aşağıdaki integralleri karşılarında yazılı bölgeler üzerinde hesaplayınız.

a) $\iint_B y dx dy, \quad B = \{(x,y) \in R^2 : \ x^2 + y^2 \le 1\}$ b) $\iint_B xy dx dy, \quad B = \{(r,\varphi) : \ r \le 4\cos\theta \text{ ve } r \le 4\sin\theta\}$

c) $\iint_B r^2 dr d\theta, \ B = \{(r,\varphi) : r \le 2\cos\theta\}$

Çözüm

6. a) $B = \{(r,\theta) : r \le 1 \text{ ve } 0 \le \theta < 2\pi\}$ ve $J = r$ olduğu gözönüne alınırsa

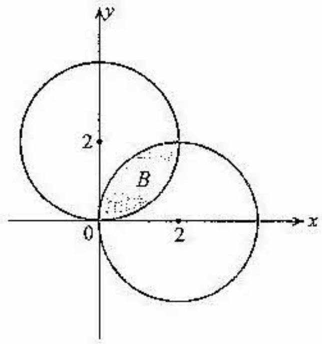


$$\iint_B 0, y dx dy = \int_0^{2\pi}\int_0^1 r\sin\theta r \, dr \, d\theta = \int_0^{2\pi}\int_0^1 r^2 \sin\theta dr d\theta$$

$$= \int_0^{2\pi} \sin\theta \frac{r^3}{3}\Bigg|_0^1 d\theta = \frac{1}{3}\int_0^{2\pi} \sin\theta d\theta$$

$$= \frac{1}{3}(-\cos\theta)\Bigg|_0^{2\pi} = 0$$

bulunur.



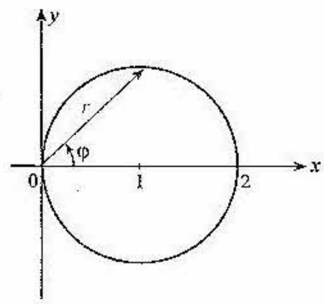
b) $\iint\limits_B xydxdy = \iint\limits_B r\cos\theta r\sin\theta r dr d\theta$

$$= \int_0^{\frac{\pi}{4}} \int_0^{2\sin\theta} r^3 \sin\theta\cos\theta dr d\theta + \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \int_0^{2\cos\theta} r^3 \sin\theta \cos\theta dr d\theta$$

$$= \int_0^{\frac{\pi}{4}} \frac{r^4}{4}\Big|_0^{2\sin\theta} \sin\theta\cos\theta d\theta + \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \frac{r^4}{4}\Big|_0^{2\cos\theta} \sin\theta\cos\theta d\theta$$

$$= 4\int_0^{\frac{\pi}{4}} \sin^5\theta\cos\theta d\theta + 4\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \cos^5\theta \sin\theta d\theta$$

$$= 4\cdot\frac{1}{6}\sin^6\theta\Big|_0^{\frac{\pi}{4}} + 4\left(-\frac{1}{6}\cos^6\theta\right)\Big|_{\frac{\pi}{4}}^{\frac{\pi}{2}} = \frac{2}{3}\left(\frac{1}{\sqrt{2}}\right)^6 + \frac{2}{3}\left(\frac{1}{\sqrt{2}}\right)^6 = \frac{1}{12} + \frac{1}{12} = \frac{1}{6}$$



c) $\int \iint\limits_B r^2 dr d\theta = \int_0^{2\pi}\int_0^{2\cos\theta} r^2 dr d\theta$

$$= \frac{1}{3}\int_0^{2\pi} r^3\Big|_0^{2\cos\theta} d\theta = \frac{8}{3}\int_0^{2\pi} \cos^3\theta d\theta$$

$$= \frac{8}{3}\int_0^{2\pi} \cos^2\theta\cos\theta d\theta = \frac{8}{3}\int_0^{2\pi} (1-\sin^2\theta)\cos\theta d\theta$$

$$= \frac{8}{3}\left(\sin\theta - \frac{1}{3}\sin^3\theta\right)\Big|_0^{2\pi} = \frac{8}{3}\cdot 0 = 0$$

$$\iint\limits_B r^2 dr d\theta = \int_0^{2\pi}\int_0^{2\cos\theta} r^2 dr d\theta = \frac{1}{3}\int_0^{2\pi} r^3\Big|_0^{2\cos\theta} d\theta = \frac{8}{3}\int_0^{2\pi}\cos^3\theta d\theta$$

$$= \frac{8}{3}\int_0^{2\pi} \cos^2\theta\cos\theta d\theta = \frac{8}{3}\int_0^{2\pi} (1-\sin^2\theta)\cos\theta d\theta$$

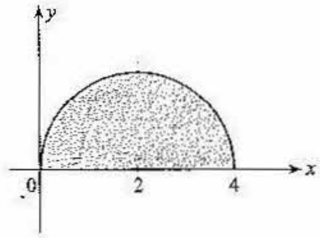
$$= \frac{8}{3}\left(\sin\theta - \frac{1}{3}\sin^3\theta\right)\Big|_0^{2\pi} = \frac{8}{3}\cdot 0 = 0$$

7. Aşağıdaki integraleri hesaplayınız.

a) $\displaystyle\int_0^4 \int_0^{\sqrt{4x-x^2}} \frac{dy\,dx}{\sqrt{x^2+y^2}}$ b) $\displaystyle\int_0^{\pi/2}\int_0^{\sin x} e^y \cos x\, dy dx$ c) $\displaystyle\int_{\pi/3}^{\pi}\int_0^{y^2} \frac{1}{y}\cos\frac{x}{y}\, dx\, dy$

Çözüm

a) $I = \int_0^4 \int_0^{\sqrt{4x-x^2}} \frac{dydx}{\sqrt{x^2+y^2}}$



$y = 0,\ y = \sqrt{4x - x^2} \Rightarrow x^2 + y^2 - 4x = 0 \Rightarrow (x-2)^2 + y^2 = 4$

Şu halde integrasyon bölgesi üst yarım dairedir.

Kutupsal koordinatlara geçilirse

$$x^2 + y^2 = 4x \Rightarrow r^2 = 4\cos\theta \Rightarrow r = 4\cos\theta$$

olacağından

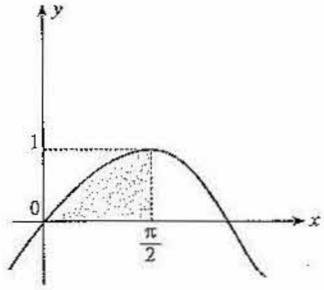
$$I = \int_0^{\frac{\pi}{2}} \int_0^{4\cos\theta} \frac{rdrd\theta}{r} = \int_0^{\frac{\pi}{2}} r\Big|_0^{4\cos\theta} d\theta = 4\int_0^{\frac{\pi}{2}} \cos\theta d\theta = 4\sin\theta\Big|_0^{\frac{\pi}{2}} = 4$$

olur.

b) $\int_0^{\frac{\pi}{2}} \int_0^{\sin x} e^y \cos x dy dx = \int_0^{\frac{\pi}{2}} e^y\Big|_0^{\sin x} \cos x dx$

$$= \int_0^{\frac{\pi}{2}} (e^{\sin x} - 1)\cos x dx = \left(e^{\sin x} - \sin x\right)\Big|_0^{\frac{\pi}{2}}$$

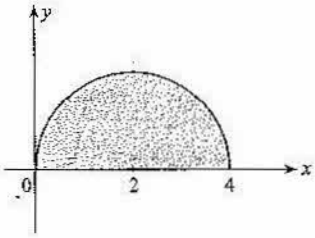
$$= e^1 - 1 - e^0 = e - 2$$



c) $\int_{\frac{\pi}{3}}^{\pi} \int_0^{y^2} \frac{1}{y}\cos\frac{x}{y} dx dy = \int_{\frac{\pi}{3}}^{\pi} \frac{1}{y} - \left(-y\sin\frac{x}{y}\right)\Big|_0^{y^2} dy = \int_{\frac{\pi}{3}}^{\pi} (-\sin y)dy = \cos y\Big|_{\frac{\pi}{3}}^{\pi} = 1 - \frac{1}{2} = -\frac{3}{2}$

Çözüm

a) $I=\int\limits_0^4\int\limits_0^{\sqrt{4x-x^2}}\frac{dydx}{\sqrt{x^2+y^2}}$



$y=0,\ y=\sqrt{4x-x^2}\Rightarrow x^2+y^2=4x=0\Rightarrow(x-2)^2+y^2=4$

Şu halde integrasyon bölgesi üst yarım dairedir.

Kutupsal koordinatlara geçilirse

$$x^2+y^2=4x\ \Rightarrow\ r^2=4\cos\theta\ \Rightarrow\ r=4\cos\theta$$

olacağından

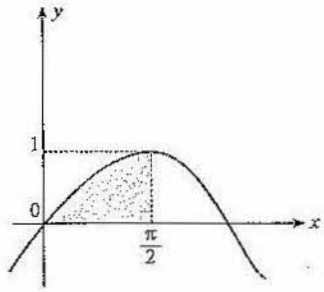
$$I=\int\limits_0^{\frac{\pi}{2}}\int\limits_0^{4\cos\theta}\frac{rdrd\theta}{r}=\int\limits_0^{\frac{\pi}{2}}r\Big|_0^{4\cos\theta}d\theta=4\int\limits_0^{\frac{\pi}{2}}\cos\theta d\theta=4\sin\theta\Big|_0^{\frac{\pi}{2}}=4$$

olur.

b) $\int\limits_0^{\frac{\pi}{2}}\int\limits_0^{\sin x}e^y\cos x dydx=\int\limits_0^{\frac{\pi}{2}}e^y\Big|_0^{\sin x}\cos x dx$

$$=\int\limits_0^{\frac{\pi}{2}}(e^{\sin x}-1)\cos x dx=\left(e^{\sin x}-\sin x\right)\Big|_0^{\frac{\pi}{2}}$$

$$=e^1-1-e^0=e-2$$

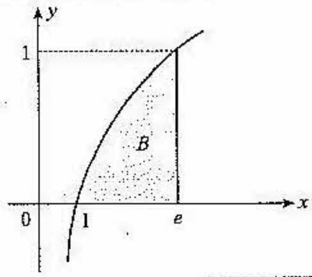


c) $\int\limits_{\frac{\pi}{3}}^{\pi}\int\limits_0^{y^2}\frac{1}{y}\cos\frac{x}{y}dxdy=\int\limits_{\frac{\pi}{3}}^{\pi}\frac{1}{y}-\left(-y\sin\frac{x}{y}\right)\Big|_0^{y^2}dy=\int\limits_{\frac{\pi}{3}}^{\pi}(-\sin y)dy=\cos y\Big|_{\frac{\pi}{3}}^{\pi}=1-\frac{1}{2}=-\frac{3}{2}$

Çözüm

$$A = \iint_B dxdy = \int_0^1 \int_{e^y}^{e} dxdy = \int_0^1 (e - e^y)dy$$

$$= ey - e^y \Big|_0^1 = 1$$

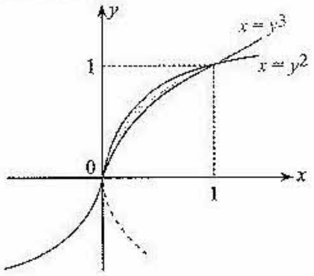


4. $x = y^3$, $x = y^2$ eğrileri arasında kalan bölgenin alanını bulunuz.

Çözüm

$$A = \iint_B dxdy = \int_0^1 \int_{y^3}^{y^2} dxdy = \int_0^1 (y^3 - y^2)dy$$

$$= \frac{-y^4}{4} + \frac{y^3}{3}\Big|_0^1 = \frac{1}{12}$$



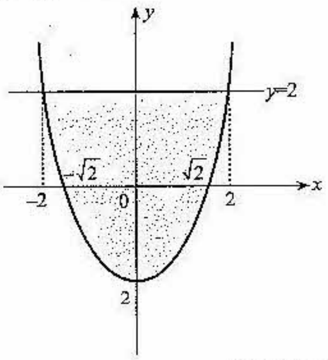
5. $y = x^2 - 2$ parabolu ile $y = 2$ doğrusu arasında kalan bölgenin alanını bulunuz.

Çözüm

$$A = \iint_B dydx = \int_{-2}^{2} \int_{x^2-2}^{2} dydx$$

$$= \int_{-2}^{2} (4 - x^2)dx = 4x - \frac{x^3}{3}\Big|_{-2}^{2}$$

$$= \frac{32}{3}$$



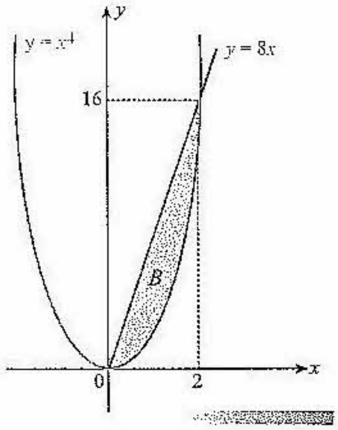
6. $y = x^4$ eğrisi $y = 8x$ doğrusu arasında kalan bölgenin alanını bulunuz.

Çözüm

$$A = \iint_B dydx = \int_0^2 \int_{x^4}^{8x} dydx$$

$$= \int_0^2 (8x - x^4)dx = 4x^2 - \frac{x^5}{5}\Bigg|_0^2$$

$$= 16 - \frac{32}{5} = \frac{48}{5}$$



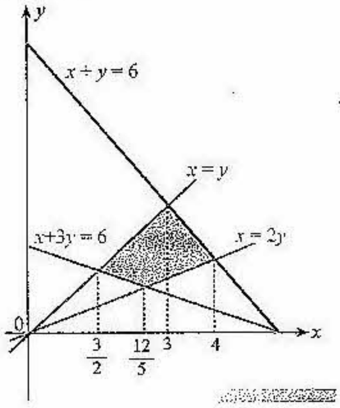
7. $x = y,\ x = 2y,\ x + y = 6,\ x + 3y = 6$ **doğruları tarafından sınırlanan bölgenin alanını bulunuz.**

Çözüm

Söz konusu doğruların grafikleri yanda verilmiştir.

$$A = \iint_B dydx = \int_{\frac{3}{2}}^{\frac{12}{5}} \int_{\frac{6-x}{5}}^{x} dydx + \int_{\frac{12}{5}}^{3} \int_{\frac{x}{2}}^{x} dydx + \int_{3}^{4} \int_{\frac{x}{2}}^{6-x} dydx$$

$$= \frac{543}{500} + \frac{81}{100} + \frac{3}{4} = \frac{1323}{500}$$



8. $y = x^2$ ve $y = 6x^2$ **parabolleri ile** $xy = 1$, $xy = 4$ **hiperbolleri tarafından sınırlanan bölgenin alanını bulunuz.**

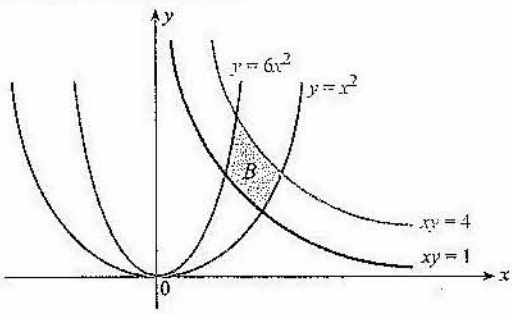
Çözüm

$$\begin{cases} \dfrac{y}{x^2} = u \\ xy = v \end{cases}$$

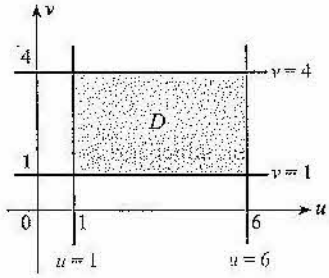
Bölge dönüşümü yapılırsa

$$J = \begin{vmatrix} x_u & x_v \\ y_u & y_v \end{vmatrix} = \frac{1}{\begin{vmatrix} u_x & u_y \\ v_x & v_y \end{vmatrix}} = \frac{1}{\begin{vmatrix} -\dfrac{2y}{x^3} & \dfrac{1}{x^2} \\ y & x \end{vmatrix}} = \frac{1}{-\dfrac{3y}{x^2}} = -\frac{1}{3u}$$

olur.



$y = x^2$ ve $y = 6x^2$ parabolleri, sırasıyla, $u = 1$ ve $u = 6$ doğrularına, $xy = 1$ ve $xy = 4$ hiperbolleride $v = 1$ ve $v = 4$ doğrularına dönüşür.



$$A = \iint_B dxdy = \iint_D \frac{1}{3u} dudv$$

$$= \frac{1}{3}\int_1^4 \int_1^6 \frac{1}{u} dudv = \frac{1}{3}\int_1^4 \ln u \Big|_1^6 dv$$

$$= \frac{4-1}{3}\ln 6 = \ln 6$$

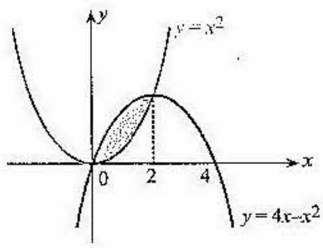
9. $y = x^2$ ve $y = 4x - x^2$ **tarafından sınırlanan bölgenin alanını bulunuz.**

Çözüm

$x^2 = 4x - x^2 \Rightarrow x^2 - 2x = 0$

$\Rightarrow x(x-2) = 0 \Rightarrow x = 0, x = 2$

$$A = \int_0^2 \int_{x^2}^{4x-x^2} dydx = \int_0^2 (4x - 2x^2)dx = 2x^2 - \frac{2}{3}x^3 \Big|_0^2 = \frac{8}{3}$$



10. $y = x^4 - 4x^2 + 4$ ve $y = x^2$ **eğrileri arasında kalan bölgenin alanını bulunuz.**

Çözüm

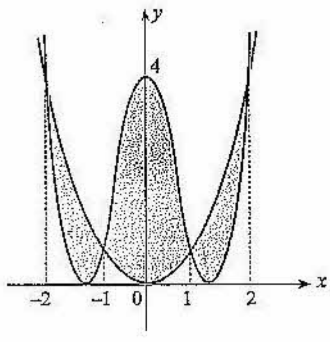
$$A = 2\int_0^1 \int_{x^2}^{x^4-4x^2+4} dydx + 2\int_1^2 \int_{x^4-4x^2+4}^{x^2} dydx$$

$$A = 2\left[\int_0^1 (x^4 - 4x^2 + 4 - x^2)dx + \int_1^2 (x^2 - x^4 + 4x^2 - 4)dx\right]$$

$$= \left[\int_0^1 (x^4 - 5x^2 + 4)dx + \int_1^2 (5x^2 - x^4 - 4)dx\right]$$

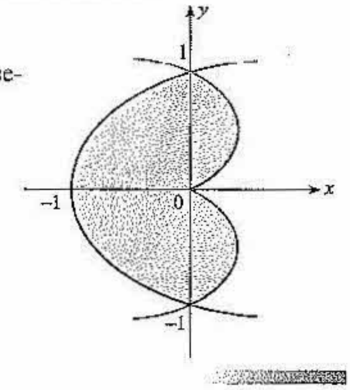
$$= 2\left[\left(\frac{x^5}{5} - \frac{5}{3}x^3 + 4x\right)\Big|_0^1 + \left(\frac{5}{3}x^3 - \frac{x^5}{5} - 4x\right)\Big|_1^2\right]$$

$$= 2\left[\frac{1}{5} - \frac{5}{3} + 4 + \frac{5}{8}\cdot 8 - \frac{32}{5} - 8 - \frac{5}{3} + \frac{1}{5} + 4\right] = 8$$



11. $x=y^2-1$ ve $x=|y|\sqrt{1-y^2}$ **eğrileri arasında kalan bölgenin alanını bulunuz.**

Çözüm



Birinci eğri parabol, ikinci eğri, y–eksenini 0, –1, 1 noktalarında kesen ve y–eksenine göre simetrik olan bir eğridir.

$$A = 2\int_0^1 \left[y\sqrt{1-y^2}-(y^2-1)\right]dy$$

$$= 2\left[-\frac{1}{3}(1-y^2)^{\frac{3}{2}}-\frac{y^3}{3}+y\right]\Bigg|_0^1$$

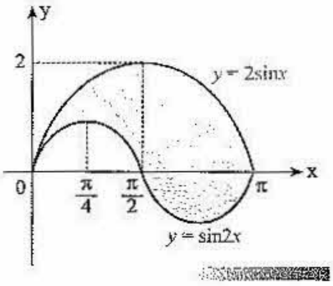
$$= 2\left[-\frac{1}{3}+1+\frac{1}{3}\right]=2$$

12. $y=2\sin x$, $y=\sin 2x$ **eğrileri ile** $x=0$, $x=\pi$ **doğruları arasında kalan bölgenin alanını bulunuz.**

Çözüm

$$A=\int_0^{\pi}\int_{\sin 2x}^{2\sin x} dydx=\int_0^{\pi}(2\sin x-\sin 2x)dx$$

$$=-2\cos x+\frac{\cos 2x}{2}\Bigg|_0^{\pi}=4\text{ br}^2$$



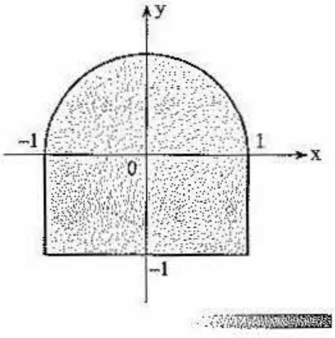
13. $y=2\sqrt{1-x^2}$ **yarım elipsi ve** $x=-1, x=1$, $y=-1$ **doğruları tarafından sınırlanan bölgenin alanını bulunuz.**

Çözüm

$$A=\int_{-1}^{1}\int_{-1}^{2\sqrt{1-x^2}} dydx=\int_{-1}^{1}\left(2\sqrt{1-x^2}+1\right)dx$$

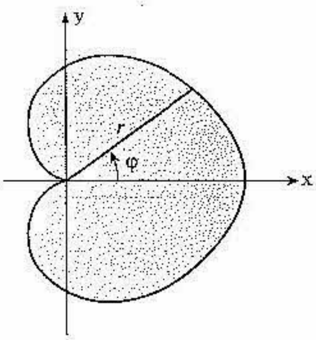
$$=2+2\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\cos\theta\cos\theta d\theta=2+2\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\frac{1+\cos 2\theta}{2}d\theta$$

$$=2+2\left(\frac{\pi}{2}+\frac{\sin 2\theta}{4}\right)\Bigg|_{-\frac{\pi}{2}}^{\frac{\pi}{2}}=2+\pi\text{ br}^2$$



14. $r = a(1+\cos\varphi)$ **kardiyoidi tarafından sınırlanan bölgenin alanını bulunuz.**

Çözüm



$$A = 2\int_0^{\frac{\pi}{2}} \int_{r=2\cos\theta}^{2(1+\cos\theta)} r dr d\theta + 2\int_0^{2(1+\cos\varphi)} r dr d\varphi$$

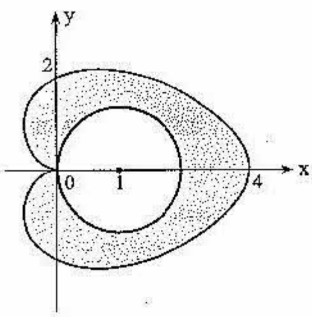
$$= \int_0^{\frac{\pi}{2}} \left[4(1+\cos\varphi)^2 - 4\cos^2\varphi\right] d\varphi + \int_{\frac{\pi}{2}}^{\pi} 4(1+\cos\varphi)^2 d\varphi$$

$$= 4\int_0^{\frac{\pi}{2}} (1+2\cos\varphi) d\varphi + 4\int_{\frac{\pi}{2}}^{\pi} \left(1+2\cos\varphi+\frac{1+\cos 2\varphi}{2}\right) d\varphi$$

$$= 4(\varphi+2\sin\varphi)\Big|_0^{\frac{\pi}{2}} + 4\left(\frac{3}{2}\varphi+2\sin\varphi+\frac{1}{4}\sin 2\varphi\right)\Big|_{\frac{\pi}{2}}^{\pi} = 4\left(\frac{\pi}{2}+2\right)+4\left(\frac{3}{2}\pi-\frac{3}{4}\pi-2\right) = 5\pi \; br^2$$

15. $r = 2(1+\cos\varphi)$ **kardiyoidi ile** $r = 2\cos\varphi$ **çemberi tarafından sınırlanan bölgenin alanını bulunuz.**

Çözüm



$$A = 2\int_0^{\frac{\pi}{2}} \int_{r=2\cos\theta}^{2(1+\cos\theta)} r dr d\theta + 2\int_{\frac{\pi}{2}}^{\pi} \int_0^{2(1+\cos\varphi)} r dr d\varphi$$

$$= \int_0^{\frac{\pi}{2}} \left[4(1+\cos\varphi)^2 - 4\cos^2\varphi\right] d\varphi + \int_{\frac{\pi}{2}}^{\pi} 4(1+\cos\varphi)^2 d\varphi$$

$$= 4\int_0^{\frac{\pi}{2}} (1+2\cos\varphi) d\varphi + 4\int_{\frac{\pi}{2}}^{\pi} \left(1+2\cos\varphi+\frac{1+\cos 2\varphi}{2}\right) d\varphi$$

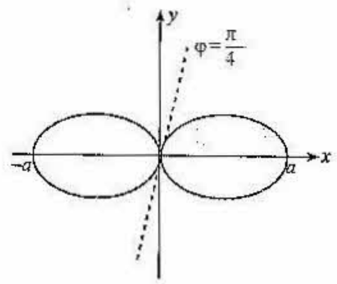
$$= 4(\varphi+2\sin\varphi)\Big|_0^{\frac{\pi}{2}} + 4\left(\frac{3}{2}\varphi+2\sin\varphi+\frac{1}{4}\sin 2\varphi\right)\Big|_{\frac{\pi}{2}}^{\pi} = 4\left(\frac{\pi}{2}+2\right)+4\left(\frac{3}{2}\pi-\frac{3}{4}\pi-2\right) = 5\pi \; br^2$$

16. $r^2 = a^2\cos^2\varphi$ lemniskatı tarafından sınırlanan bölgenin alanını bulunuz.

Çözüm

$$A = 4\iint_B r dr d\varphi = 4\int_{\theta=0}^{\frac{\pi}{4}}\int_{r=0}^{a\sqrt{\cos 2\theta}} r dr d\varphi$$

$$= 2\int_0^{\frac{\pi}{4}} a^2\cos 2\varphi d\varphi = 2.a^2\frac{\sin 2\varphi}{2}\Big|_0^{\frac{\pi}{4}} = a^2$$



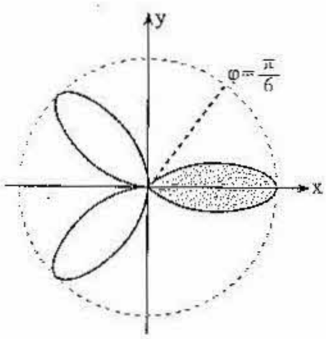
17. $r = 12\cos 3\varphi$ gülünün bir yaprağının alanını bulunuz.

Çözüm

$$A = 2\int_{\theta=0}^{\frac{\pi}{6}}\int_{r=0}^{12\cos 3\varphi} r dr d\varphi = a\int_0^{\frac{\pi}{6}} r^2\Big|_0^{12\cos 3\varphi} d\varphi$$

$$\overset{\text{İYİ}}{=} \int_0^{\frac{\pi}{6}} 12^2\left(\frac{\cos 6\varphi + 1}{2}\right)d\varphi = 72\left(\frac{\sin 6\varphi}{6} + \varphi\right)\Big|_0^{\frac{\pi}{6}} = 12\pi$$

birimkare olur.



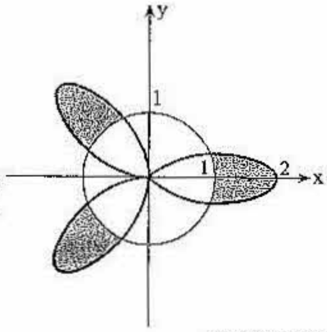
18. $r = 1 + \cos 3\varphi$ gülünün içinde, $r = 1$ çemberinin dışında bulunan bölgenin alanını bulunuz.

Çözüm

$$A = 3.2\int_{\theta=0}^{\frac{\pi}{6}}\int_{r=1}^{1+\cos 3\varphi} r dr d\varphi = 3\int_0^{\frac{\pi}{6}}(2\cos 3\varphi + \cos^2 3\varphi)d\varphi$$

$$= 3\int_0^{\frac{\pi}{6}}\left(2\cos 3\varphi + \frac{\cos 6\varphi + 1}{2}\right)d\varphi = 3\left(\frac{2\sin 3\varphi}{3} + \frac{\sin 6\varphi}{12} + \frac{\varphi}{2}\right)\Big|_0^{\frac{\pi}{6}} = 2 + \frac{\pi}{4}$$

birimkaredir.



19. $r = 1 + \cos\varphi$ ve $r = 1 - \cos\varphi$ kardioidlerinin her ikisinin de içinde kalan ortak bölgenin alanını bulunuz.

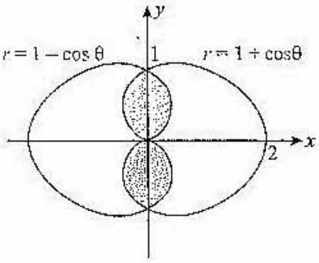
Çözüm

$$A = 4\int_0^{\frac{\pi}{2}}\int_{r=0}^{1-\cos\theta} r\,dr\,d\theta = 2\int_0^{\frac{\pi}{2}}(1 - 2\cos\theta + \cos^2\theta)\,d\theta$$

$$= 2\int_0^{\frac{\pi}{2}}\left(1 - 2\cos\theta + \frac{\cos 2\theta + 1}{2}\right)d\theta$$

$$= 2\left(\theta - 2\sin\theta + \frac{\sin 2\theta}{4} + \frac{\theta}{2}\right)\Bigg|_0^{\frac{\pi}{2}} = \frac{3\pi}{2} - 4$$

birimkaredir.



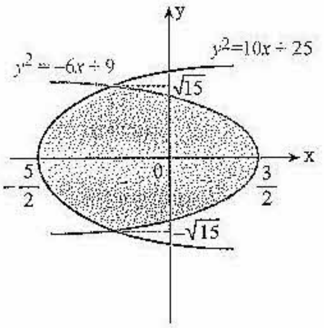
20. $y^2 = 10x + 25$ ve $y^2 = -6x + 9$ parabolleri tarafından sınırlanan bölgenin alanını bulunuz.

Çözüm

$$\frac{y^2 - 25}{10} = \frac{9 - y^2}{6} \Rightarrow y = \mp\sqrt{15}$$

$$A = \int_{-\sqrt{15}}^{\sqrt{15}}\int_{\frac{y^2-25}{10}}^{\frac{9-y^2}{6}} dx\,dy = \int_{-\sqrt{15}}^{\sqrt{15}}\left(\frac{9 - y^2}{6} - \frac{y^2 - 25}{10}\right)dy$$

$$= \int_{-\sqrt{15}}^{\sqrt{15}}\left(4 - \frac{4y^2}{15}\right)dy = \left(4y - \frac{4y^3}{45}\right)\Bigg|_{-\sqrt{15}}^{\sqrt{15}} = \frac{16}{3}\sqrt{15}$$



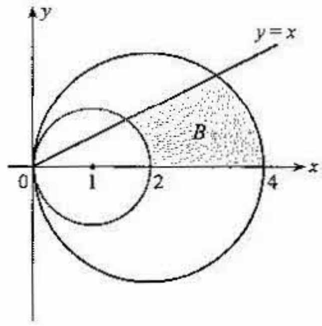
21. $x^2 + y^2 = 2x$, $x^2 + y^2 = 4x$ çemberleri ile $y = 0$, $y = x$ doğruları arasında kalan bölgenin alanını bulunuz.

Çözüm

$x^2+y^2=2x \Rightarrow r=2\cos\varphi$

$x^2+y^2=4x \Rightarrow r=4\cos\varphi$

$y=x \Rightarrow \varphi=\dfrac{\pi}{4}$



$$A=\iint_B rdrd\varphi=\int_{\theta=0}^{\frac{\pi}{4}}\int_{r=2\cos\varphi}^{4\cos\varphi} rdrd\varphi=\int_0^{\pi/4}\frac{r^2}{2}\bigg|_{2\cos\varphi}^{4\cos\varphi}d\varphi=\frac{1}{2}\int_0^{\frac{\pi}{4}}12\cos^2\varphi d\varphi$$

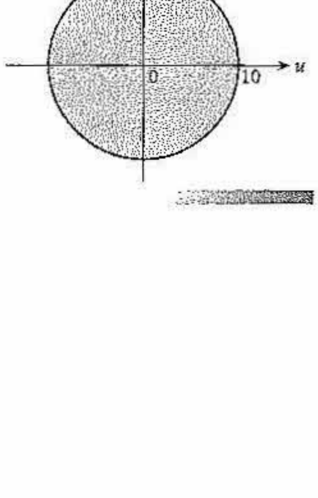
$$=3\int_0^{\pi/4}(1+\cos 2\varphi)d\varphi=3\left(\frac{\sin 2\varphi}{2}+\varphi\right)\bigg|_0^{\frac{\pi}{4}}=\frac{3\pi}{4}+\frac{3}{2}\ br^2$$

22. $(x-2y+3)^2+(3x+4y-1)^2=100$ **elipsi tarafından sınırlanan bölgenin alanını bulunuz.**

Çözüm

$\begin{cases} u=x-2y+3 \\ v=3x+4y-1 \end{cases}$ dönüşümü yapılırsa elipsin görüntüsü

$u^2+v^2=10^2$ çemberi olur.



$$J=\frac{\partial(x,y)}{\partial(u,v)}=\frac{1}{\frac{\partial(u,v)}{\partial(x,y)}}=\frac{1}{\begin{vmatrix}1 & -2\\ 3 & 4\end{vmatrix}}=\frac{1}{10}$$ olduğundan

$$A=\iint_D |J|\,dudv=\iint_D \frac{1}{10}\,dudv=\frac{1}{10}\int_{\theta=0}^{2\pi}\int_{r=0}^{10} rdrd\theta$$

$$=\frac{1}{10}\cdot\frac{100}{2}.2\pi=10\pi$$

birim kare olur.

23. $z = 2x^2 + y^2 + 1$ paraboloidi, $x + y = 1$ düzlemi ve koordinat düzlemleri tarafından sınırlanan bölgenin hacmini bulunuz.

Çözüm

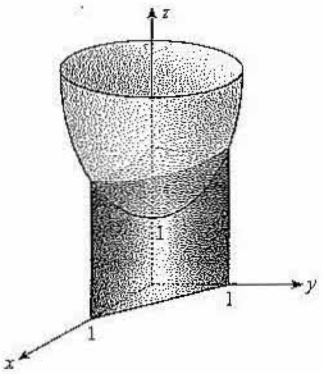
$$V = \int_0^1 \int_0^{1-x} (2x^2 + y^2 + 1)dydx$$

$$= \int_0^1 2x^2y + \frac{y^3}{3} + y\Big|_0^{1-x} dx$$

$$= \int_0^1 \left[2x^2 - 2x^3 + \frac{(1-x)^3}{3} + 1 - x\right]dx$$

$$= \frac{2}{3}x^3 - \frac{1}{2}x^4 - \frac{(1-x)^4}{12} + x - \frac{x^2}{2}\Big|_0^1$$

$$= \frac{2}{3} - \frac{1}{2} + \frac{1}{2} + \frac{1}{12} = \frac{3}{4}$$



24. $\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 + \left(\frac{z}{c}\right)^2 = 1$ elipsoidi tarafından sınırlanan bölgenin hacmini bulunuz.

Çözüm

$$V = \iint_B zdxdy = 2\iint_B c\sqrt{\frac{x^2}{a^2} + \frac{y^2}{b^2}}dxdy$$

integralinde

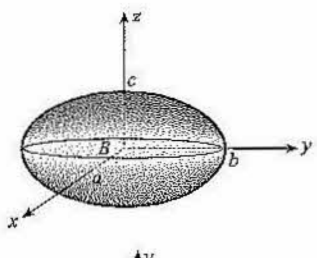
$$\begin{cases} x = ar\cos\varphi \\ y = br\sin\varphi \end{cases}$$

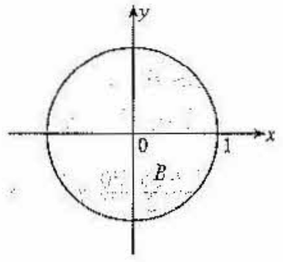
dönüşümü yapılırsa, $J = abr$ olacağından,

$$V = 2abc\iint_D r^2 drd\varphi = 2abc\int_0^{2\pi}\int_0^1 r^2 drd\varphi$$

$$= 2abc\int_0^{2\pi} \frac{r^3}{3}\Big|_0^1 d\varphi = \frac{2}{3}abc\int_0^{2\pi} d\varphi = \frac{4}{3}\pi abc$$

birimküp bulunur.





25. $z=x^2+y^2$ paraboloidi ile $z=0$, $x=-3$, $y=-3$, $y=3$ düzlemleri tarafından sınırlanan bölgenin hacmini bulunuz.

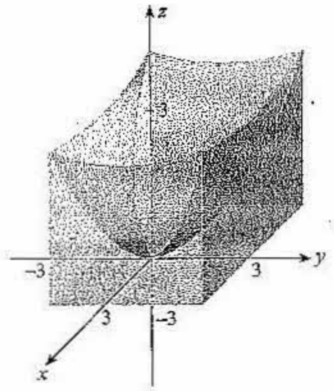
Çözüm

$$V=\int_{-3}^{3}\int_{-3}^{3}(x^2+y^2)dydx=\int_{-3}^{3}x^2y+\frac{y^3}{3}\Big|_{-3}^{3}dx$$

$$=\int_{-3}^{3}(3x^2+9+3x^2+9)dx=2x^3+18x\Big|_{-3}^{3}$$

$$=2(54+54)=216$$

birimküp olur.



26. xy düzlemi, $4z=x^2+y^2$ paraboloidi ve $x^2+y^2=4$ silindiri tarafından sınırlanan bölgenin hacmini bulunuz.

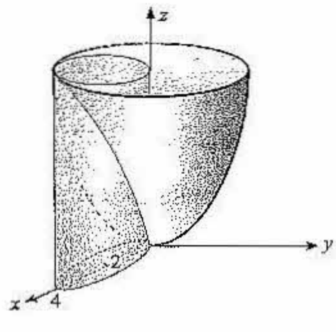
Çözüm

$$V=\iint\limits_{x^2+y^2=4x}\frac{x^2+y^2}{4}dydx=2\int_{\theta=0}^{\frac{\pi}{2}}\int_{r=0}^{4\cos\theta}\frac{r^2}{4}drd\theta$$

$$=2\int_{\theta=0}^{\frac{\pi}{2}}\int_{r=0}^{4\cos\theta}\frac{r^3}{4}drd\theta=\int_{0}^{\frac{\pi}{2}}32(\cos\theta)^4d\theta$$

$$=32\int_{0}^{\frac{\pi}{2}}(\cos^2\theta-\cos^2\theta.\sin^2\theta)d\theta$$

$$=32\int_{0}^{\frac{\pi}{2}}\left[\frac{1+\cos 2\theta}{2}-\frac{(1-\cos 4\theta)}{8}\right]d\theta=6\pi$$



27. $x+y+z=3$, $x^2+y^2=1$, $z=0$ yüzeyleri tarafından sınırlanan bölgenin hacmini bulunuz.

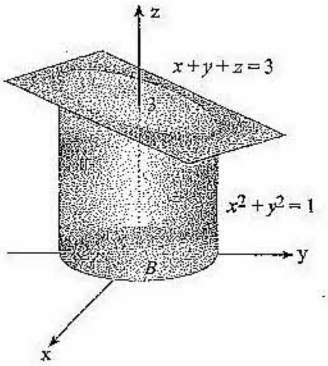
Çözüm

$$V=\iint_B (3-x-y)\,dxdy$$

$$=\int_{\varphi=0}^{2\pi}\int_{r=0}^{1}(3r-r^2\cos\varphi-r^2\sin\varphi)\Big|_0^1\,d\varphi$$

$$=\int_0^{2\pi}\left(\frac{3}{2}-\frac{\cos\varphi}{3}-\frac{\sin\varphi}{3}\right)d\varphi=\frac{3}{2}\varphi-\frac{\sin\varphi}{3}+\frac{\cos\varphi}{3}\Big|_0^{2\pi}$$

$$=3\pi+\frac{1}{3}-\frac{1}{3}=3\pi$$

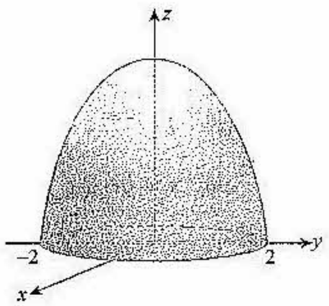


28. $2z=4-x^2-y^2$ paraboloidi ile $z=0$ düzlemi tarafından sınırlanan bölgenin hacmini bulunuz.

Çözüm

$$V=\iint_{x^2+y^2=4}\frac{4-x^2-y^2}{2}dzdydx=\int_{\theta=0}^{2\pi}\int_{r=0}^{2}\frac{4-r^2}{2}r\,drd\varphi$$

$$=\int_0^{2\pi}\int_0^{2}\left(2r-\frac{r^3}{2}\right)drd\varphi=\int_0^{2}r^2-\frac{r^4}{8}\Big|_0^2\,d\varphi=2\pi(2)=4\pi$$



29. Üstten $z=x^2+y^2$ paraboloidi, alttan xy düzlemi ve yandan $x^2+y^2=4$ silindiri tarafından sınırlanan bölgenin hacmini bulunuz.

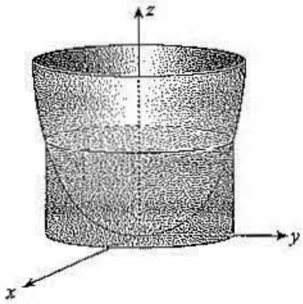
Çözüm

$$V = \iint_B (x^2 + y^2)dydx$$

$$= \int_0^{2\pi} \int_0^2 r^3 dr d\varphi = \int_0^{2\pi} \frac{r^4}{4}\Big|_0^2 d\varphi$$

$$= 4\int_0^{2\pi} d\varphi = 8\pi$$

birimküp olur.



30. xy düzlemi $z=e^{-(x^2+y^2)}$ yüzeyi ve $x^2+y^2=1$ silindiri tarafından sınırlanan bölgenin hacmini bulunuz.

Çözüm

$$V = \iint_B e^{-(x^2+y^2)}dydx = \int_{\varphi=0}^{2\pi} \int_{r=0}^{1} e^{-r^2} r dr d\varphi = -\frac{1}{2}2\pi e^{-r^2}\Big|_0^1 = -\pi(e^{-1}-1) = \pi(1-e^{-1})$$

31. $a, b > 0$ dır. xy düzlemi, $z = \frac{x^2}{a^2} + \frac{y^2}{b^2}$ paraboloidi ve $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 2 \cdot \frac{x}{a}$ silindiri arasında kalan bölgenin hacmini bulunuz.

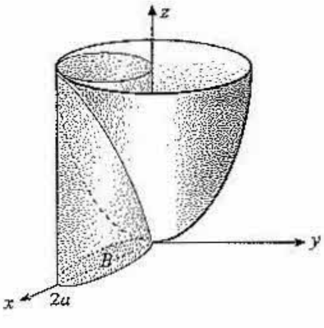
Çözüm

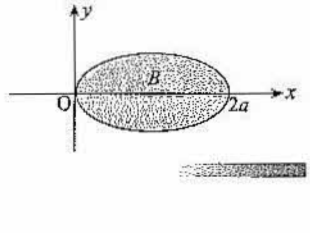
$V = \iint_B \left(\frac{x^2}{a^2} + \frac{y^2}{b^2}\right)dydx$ olur.

$x = ar\cos\varphi,\ \ y = br\sin\varphi$ dönüşümü yapılırsa,

$$V = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_{r=0}^{2\cos\varphi} r^2.abrdrd\varphi = \frac{ab}{4}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (2\cos\varphi)^4 d\varphi$$

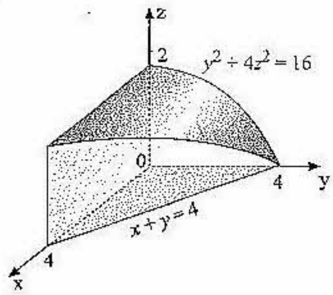
$$= 8ab\int_0^{\frac{\pi}{2}} \cos^4\varphi d\varphi = 8ab\frac{3\pi}{16} = \frac{3}{2}\pi ab \text{ olur.}$$





32. Birinci bölgede, $x = 0$, $y = 0$, $z = 0$, $x + y = 4$ düzlemleriyle $y^2 + 4z^2 = 16$ silindiri tarafından sınırlanan bölgenin hacmini bulunuz.

Çözüm



$$V = \int_0^4 \int_0^{4-y} \frac{1}{2}\sqrt{16-y^2}\,dxdy = \frac{1}{2}\int_0^{44} \sqrt{16-y^2}\,dy$$

$$= \frac{1}{2}\int_0^{\frac{\pi}{2}} (4 - 4\sin t)\sqrt{16 - 16\sin^2 t}\,.4\cos t dt$$

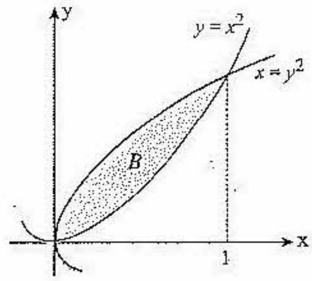
$$= 32\int_0^{\frac{\pi}{2}} (\cos^2 t - \cos^2 t \sin t)dt$$

$$= 32\int_0^{\frac{\pi}{2}} \left(\frac{1+\cos 2t}{2} - \cos^2 t \sin t\right)dt = 32\left(\frac{t}{2} + \frac{1}{4}\sin 2t + \frac{1}{3}\cos^3 t\right)\Big|_0^{\frac{\pi}{2}} = 8\pi - \frac{32}{3}$$

birimküp olur.

33. $y = x^2$ ve $x = y^2$ parabolleri tarafından sınırlanan bölgeye yerleştirilen bir levhanın yoğunluğu, her noktada o noktanın 0x–eksenine olan uzaklığının karesi ile orantılı olarak değişmektedir. Bu levhanın kütlesini bulunuz.

Çözüm



$\sigma(x,y) = ky^2$ dir (k, orantı katsayısıdır.)

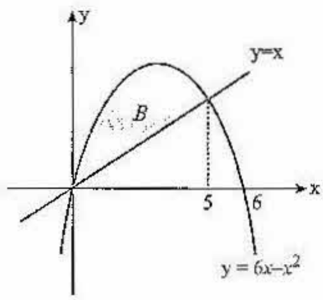
$$M = \iint_B \sigma(x,y)dydx = \int_0^1 \int_{x^2}^{\sqrt{x}} ky^2\,dydx$$

$$= \int_0^1 \frac{k}{3}\left(x^{\frac{3}{2}} - x^6\right)dx = \frac{k}{3}\left(\frac{2}{5}x^{5/2} - \frac{1}{7}x^7\right)\Big|_0^1 = \frac{3}{35}k$$

34. $y = 6x - x^2$ parabolü ile $y = x$ doğrusu tarafından sınırlanan bölgeye yerleştirilen homogen levhanın ağırlık merkezini bulunuz.

Çözüm

$\sigma(x,y) = k, \quad \overline{x} = \dfrac{M_x}{M}, \quad \overline{y} = \dfrac{M_y}{M}$



$$M_x = \iint_B x\sigma(x,y)\,dydx = \int_0^5 \int_x^{6x-x^2} xk\,dydx = k\int_0^5 (5x^2 - x^3)dx = \frac{625k}{12}$$

$$M = \iint_B \sigma(x,y)\,dydx = \int_0^5 \int_x^{6x-x^2} k\,dydx = \frac{125k}{6} \Rightarrow \overline{x} = \frac{M_x}{M} = \frac{\frac{625}{12}}{\frac{125}{6}} = \frac{5}{2}$$

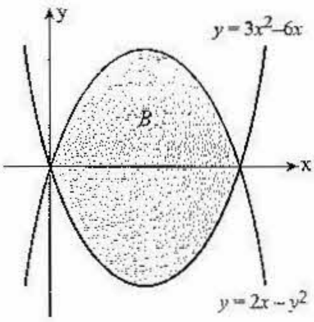
Benzer şekilde $\overline{y} = 5$ bulunur. Ağırlık merkezi $\left(\dfrac{5}{2}, 5\right)$ noktasıdır.

35. $y = 2x - x^2$ ve $y = 3x^2 - 6x$ parabolleri tarafından sınırlanan homogen levhanın ağırlık merkezini bulunuz.

Çözüm

$\sigma(x,y) = k, \quad G(\overline{x}, \overline{y})$, ağırlık merkezi olsun.

$\overline{x} = \dfrac{M_x}{M}, \quad \overline{y} = \dfrac{M_y}{M}$ dir.



$$M_x = \iint_B x\sigma(x,y)\,dydx = \int_0^2 \int_{3x^2-6x}^{2x-x^2} kx\,dydx = \int_0^2 kxy\Big|_{3x^2-6x}^{2x-x^2} dx$$

$$= k\int_0^2 (8x^2 - 4x^3)dx = k\left(\frac{8}{3}x^3 - x^4\right)\Big|_0^2 = k\left(\frac{64}{3} - \frac{48}{3}\right) = \frac{16}{3}k$$

$$M = \iint_B \sigma(x,y)\,dydx = k\int_0^2 \int_{3x^2-6x}^{2x-x^2} dydx = k\int_0^2 (8x - 4x^2)dx = k\left(4x^2 - \frac{4}{3}x^3\right)\Big|_0^2 = k\left(\frac{48}{3} - \frac{32}{3}\right) = \frac{16}{3}k$$

$$\overline{x} = \frac{M_x}{M} = \frac{\frac{16}{3}k}{\frac{16}{3}k} = 1$$

$$M_y = \int_0^2 \int_{3x^2-6x}^{2x-x^2} kydydx = k\int_0^2 \left[\frac{(2x-x^2)^2}{2} - \frac{(3x^2-6x)^2}{2}\right]dx$$

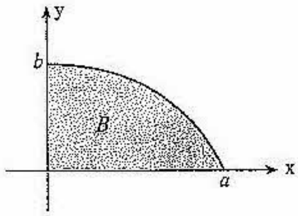
$$= \frac{k}{2}\int_0^2 (4x^2 - 4x^3 + x^4 - 9x^4 + 36x^3 - 36x^2)$$

$$= k\int_0^2 (-16x^2 + 16x^3 - 4x^4)dx = k\left(-\frac{16}{3}x^3 + 4x^4 - \frac{4}{5}x^5\Big|_0^2\right) = -\frac{64}{15}k$$

$$\overline{y} = \frac{M_y}{M} = -\frac{\frac{64}{15}}{\frac{16}{3}} = -\frac{4}{5}$$

36. Birinci bölgede, $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$ elipsi ile koordinat eksenleri arasında kalan bölgeye yerleştirilen homogen levhanın ağırlık merkezini bulunuz.

Çözüm



$G(\overline{x},\overline{y})$ $\sigma(x,y) = k$ ve levhanın ağırlık merkezi $G(\overline{x},\overline{y})$ olsun.

$$\begin{cases} \dfrac{x}{a} = r\cos\varphi \\ \dfrac{y}{b} = r\sin\varphi \end{cases}$$ dönüşümü için $J = a.b.r$ olur.

$$M_x = \iint_B x\sigma(x,y)dydx = \int_{\varphi=0}^{\frac{\pi}{2}} \int_{r=0}^{1} a.r\cos\varphi.k.abr\,drd\varphi = k\frac{a^2b}{3}\int_0^{\frac{\pi}{2}} \cos\varphi\, d\varphi = k\frac{a^2b}{3}$$

$$M = \iint_B \sigma(x,y)dydx = kab\int_{\theta=0}^{\frac{\pi}{2}} \int_{r=0}^{1} r.drd\varphi = k\frac{a.b}{2}\frac{\pi}{2} = k\frac{ab\pi}{4}$$

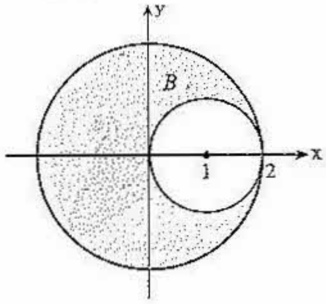
$$\overline{x} = \frac{M_x}{M} = \frac{4a}{3\pi}$$

$$M_y = \int_{\theta=0}^{\frac{\pi}{2}} \int_{r=0}^{1} br\sin\varphi.kab\,rdrd\varphi = k\frac{ab^2}{3}$$

$$\overline{y} = \frac{M_y}{M} = \frac{4b}{3\pi}$$

37. $x^2+y^2=4$ ve $(x-1)^2+y^2=1$ çemberleri tarafından sınırlanan bölgeye yerleştirilen homogen levhanın ağırlık merkezini bulunuz.

Çözüm



Bölge x eksenine göre simetrik ve cisim homogen olduğundan $\bar{y}=0$ dır.

$(x-1)^2+y^2=1 \Rightarrow r=2\cos\varphi$

$x^2+y^2=4 \Rightarrow r=2$

$$M_x=\iint_B x\sigma(x,y)dydx$$

$$=2\int_0^{\frac{\pi}{2}}\int_{2\cos\varphi}^{2} r\cos\varphi kr\,drd\varphi+2\int_{\varphi=\frac{\pi}{2}}^{\pi}\int_{r=0}^{2} r\cos\varphi k.rdrd\varphi$$

$$=\frac{2}{3}k\int_0^{\frac{\pi}{2}}\cos\varphi(8-8\cos^3\varphi)d\varphi+\frac{2}{3}k\int_{\frac{\pi}{2}}^{\pi}8\cos\varphi\,d\varphi$$

$$=\frac{16}{3}k\left(\int_0^{\frac{\pi}{2}}(\cos\varphi-\cos^2\varphi+\cos^2\varphi\sin^2\theta)d\varphi+\int_{\frac{\pi}{2}}^{\pi}\cos\varphi d\varphi\right)$$

$$=\frac{16}{3}k\left(\sin\varphi-\left(\frac{\sin 2\varphi}{4}+\frac{\varphi}{2}\right)+\left(\frac{\varphi}{8}-\frac{\sin 4\varphi}{32}\right)\Big|_0^{\frac{\pi}{2}}+\sin\varphi\Big|_{\frac{\pi}{2}}^{\pi}\right)$$

$$=\frac{16}{3}k\left(1-\frac{\pi}{4}+\frac{\pi}{16}-1\right)=\frac{16}{3}k\left(-\frac{3}{16}\pi\right)=-k\pi$$

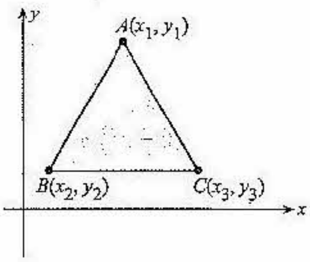
$$M=\iint_B\sigma(x,y)dydx=2\int_{\theta=0}^{\frac{\pi}{2}}\int_{r=2\cos\theta}^{2}krdrd\varphi+2\int_{\frac{\pi}{2}}^{\pi}\int_{r=0}^{2}krdrd\varphi$$

$$=k\int_0^{\frac{\pi}{2}}(4-4\cos^2\varphi)d\varphi+\int_{\frac{\pi}{2}}^{\pi}4kd\varphi=4k\left(\varphi-\left(\frac{\varphi}{2}+\frac{\sin 2\varphi}{4}\right)\Big|_0^{\frac{\pi}{2}}+2\pi k\right)=4k\left(\frac{3\pi}{4}\right)=3\pi k$$

$$\bar{x}=\frac{M_x}{M}=-\frac{k\pi}{3\pi k}=-\frac{1}{3}$$

38. Köşeleri $A(x_1, y_1)$, $B(x_2, y_2)$, $C(x_3, y_3)$ olan üçgensel levhanın ağırlık merkezini bulunuz.

Çözüm



AB, BC ve AC doğrularının denklemleri, sırası ile,

$$y = y_1 + m_1(x - x_1), \quad \left(m_1 = \frac{y_2 - y_1}{x_2 - x_1}\right)$$

$$y = y_2 + m_2(x - x_1), \quad \left(m_2 = \frac{y_3 - y_2}{x_3 - x_2}\right)$$

$$y = y_1 + m_3(x - x_2), \quad \left(m_3 = \frac{y_3 - y_1}{x_3 - x_1}\right)$$

olur. Gerekli işlemler yapılarak

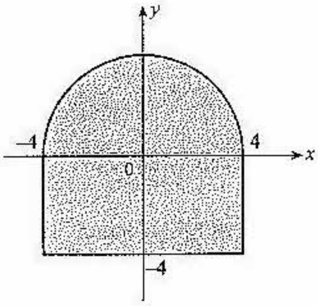
$$A = \iint_B dxdy = x_1(y_2 - y_3) + x_2(y_3 - y_1) + x_3(y_1 - y_2)$$

$$\overline{x} = \frac{x_1 + x_2 + x_3}{3}, \quad \overline{y} = \frac{y_1 + y_2 + y_3}{3}$$

olduğu görülebilir.

39. $y = \sqrt{16 - x^2}$ yarı çemberi ile $x = -4$, $x = 4$, $y = -4$ doğruları tarafından sınırlanan bölgeye yerleştirilen ve (x, y) noktasındaki yoğunluğu $\sigma(x, y) = 4 + y$ olan levhanın ağırlık merkezini bulunuz.

Çözüm



$$M_x = \iint_B x\sigma(x,y)\,dydx = \int_{-4}^{\sqrt{16-x^2}} x(4 + y)\,dydx$$

$$= \int_{-4}^{4} 4xy + \frac{xy^2}{2}\Bigg|_{-4}^{\sqrt{16-x^2}} dx$$

$$= \int_{-4}^{4} \left[4x\sqrt{16 - x^2} + \frac{x}{2}(16 - x^2) + 16x - 8x\right]dx$$

$$= -2\frac{2}{3}(16 - x^2)^{\frac{3}{2}} - \frac{x^4}{8} + 8x^2\Bigg|_{-4}^{4} = -\frac{4^4}{8} + 32 + \frac{4^4}{8} - 32 = 0 \Rightarrow \overline{x} = 0 \text{ olur.}$$

$$M = \int_{-4}^{4}\int_{-4}^{\sqrt{16-x^2}} (4+y)dydx + \int_{-4}^{4} 4y + \frac{y^2}{2}\Bigg|_{-4}^{\sqrt{16-x^2}} dx = \int_{-4}^{4}\left(4\sqrt{16-x^2} + \frac{16-x^2}{2} + 16 - 8\right)dx$$

$$= \frac{32}{3}(10+3\pi)$$

$$M_y = \int_{-4}^{4}\int_{-4}^{\sqrt{16-x^2}} y(4+y)\,dy\,dx = \int_{-4}^{4}\left(2y^2+\frac{y^3}{3}\right)\Bigg|_{-4}^{\sqrt{16-x^2}} dx$$

$$= \int_{-4}^{4}\left[2(16-x^2)+\frac{1}{3}(16-x^2)^{3/2}-\frac{32}{3}\right]dx = \frac{32}{3}(8+3\pi)$$

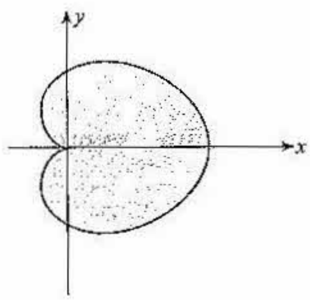
olur. Buna göre,

$$\overline{y} = \frac{M_y}{M} = \frac{\frac{32}{3}(10+3\pi)}{\frac{32}{3}(8+3\pi)} = \frac{10+3\pi}{8+3\pi}$$

olacaktır.

40. $r = a(1+\cos\varphi)$ kardiyoidi tarafından sınırlanan bölgeye yerleştirilen homogen levhanın ağırlık merkezini bulunuz.

Çözüm



$$M_x = \int_0^{2\pi}\int_{r=0}^{a(1+\cos\theta)} r\cos\theta krdrd\theta = \frac{k}{3}\int_0^{2\pi} a^3(1+\cos\theta)^3.\cos\theta d\theta$$

$$= \frac{ka^3}{3}\int_0^{2\pi}(\cos\theta+3\cos^2\theta+3\cos^3\theta+\cos^4\theta)d\theta = \frac{5}{4}k\pi a^3$$

$$M = \int_0^{2\pi}\int_{r=0}^{a(1+\cos\theta)} krdrd\theta = k\frac{a^2}{2}\int_0^{2\pi}\left[1+2\cos\theta+\frac{1+\cos 2\theta}{2}\right]d\theta$$

$$= k\frac{a^2}{2}\left(\frac{3\theta}{2}+2\sin\theta+\frac{\sin 2\theta}{4}\Bigg|_0^{2\pi}\right) = \frac{3}{2}k\pi a^2 \text{ olur.}$$

$\overline{x} = \frac{M_x}{M} = \frac{5a}{6}$ bulunur. Simetriden $\overline{y} = 0$ olacağı açıktır.

Buna göre ağırlık merkezi $\left(\frac{5a}{6}, 0\right)$ noktasıdır.

BÖLÜM PROBLEMLERİ VE ÇÖZÜMLERİ

1. Hangi B bölgesi üzerinde

$$\iint_B (4-x^2-2y^2)dxdy$$

integrali maksimum değerini alır? Bu değeri bulunuz.

Çözüm

İntegralin maksimum değer alması için $4-x^2-2y^2 \geq 0 \Rightarrow x^2+2y^2 \leq 4$ olmalıdır.

$$\left.\begin{array}{l} \dfrac{x}{2} = r\cos\theta \\ \dfrac{y}{\sqrt{2}} = r\sin\theta \end{array}\right\} \Rightarrow J = 2\sqrt{2}\,r \ \ olur.$$

$$\iint_{x^2+2y^2\leq 4} (4-x^2-2y^2)dydx = \int_{\varphi=0}^{2\pi}\int_{r=0}^{1}(4-4r^2)2\sqrt{2}\,drd\varphi$$

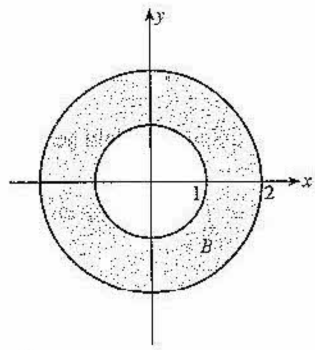
$$= 8\sqrt{2}\int_{\varphi=0}^{2\pi}\left(\frac{r^2}{2}-\frac{r^4}{4}\right)\Bigg|_0^1 d\varphi = 8\sqrt{2}\left(\frac{1}{2}-\frac{1}{4}\right)2\pi = 4\sqrt{2}\,\pi$$

2. Hangi B bölgesi üzerinde

$$\iint_B \left[(x^2+y^2)^2-5(x^2+y^2)+4\right]dxdy$$

integrali minimum değerini alır? Bu değeri bulunuz.

Çözüm



$x^2+y^2=z$ denirse integrant z^2-5z+4 olur. İntegralin minimum değerini alması için $z^2-5z+4 \leq 0$ olmalıdır. O halde $1 \leq z \leq 4$ dolayısıyla $1 \leq x^2+y^2 \leq 4$ olmalıdır. Şu halde integral yarıçapı 1 ve 2 birim olan çemberler arasında kalan halka olmalıdır.

Buna göre integralin alabileceği minimum değer

$$\iint_B \left[(x^2+y^2)^2-5(x^2+y^2)+4\right]dxdy = \int_0^{2\pi}\int_0^{1}(r^4-5r^2+4)rdrd\varphi$$

$$= \int_0^{2\pi}\left(\frac{r^6}{6}-\frac{5}{4}r^4+2r^2\right)\Bigg|_1^2 d\varphi = -\frac{9}{4}.2\pi = -\frac{9}{2}\pi$$

olur.

3. $\int_0^2 (\arctan \pi x - \arctan x)dx$ integralini hesaplayınız. (Önce integrantı bir integral formunda yazınız.)

Çözüm

$$I = \int_0^2 (\arctan \pi x - \arctan x)dx = \int_0^2 \int_1^{\pi} \frac{x}{1+(xt)^2} dt dx = \int_1^{\pi} \frac{1}{2t^2} \int_0^2 \frac{2t^2 x}{1+x^2t^2} dx dt$$

$$= \int_1^{\pi} \frac{1}{2t^2} \ln(1+x^2t^2)\Big|_0^2 dt = \int_1^{\pi} \frac{\ln(1+4t^2)}{2t^2} dt = -\frac{1}{2t}\ln(1+4t^2)\Big|_1^{\pi} + \int_1^{\pi} \left(\frac{1}{2t}\right) \frac{8t}{1+4t^2} dt$$

$$= \frac{\ln 5}{2} - \frac{\ln(1+4\pi^2)}{2\pi} + \frac{1}{2}\int_1^{\pi} \frac{8\,dt}{1+4t^2} = \frac{\ln 5}{2} - \frac{\ln(1+4\pi^2)}{2\pi} + \frac{1}{2}\arctan 2t\Big|_1^{\pi}$$

$$= \frac{\ln 5}{2} - \frac{\ln(1+4\pi^2)}{2\pi} - \frac{1}{2}\arctan 2$$

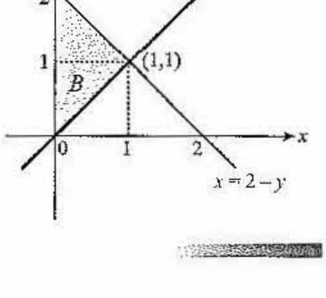
4. Tabanı xy düzleminde B olan bir dik silindir (dairesel olmayan) üstten $z = x^2 + y^2$ paraboloidi ile kapatılıyor. Bu silindirin hacmi

$$V = \int_0^1 \int_0^y (x^2+y^2)dxdy + \int_1^2 \int_0^{2-y} (x^2+y^2)dxdy$$

olduğuna göre B bölgesini xy düzleminde gösteriniz. İntegrasyon sırasını değiştirerek yazınız. Hacmi hesaplayınız.

Çözüm

4. B bölgesi yandaki şekilde renkli olarak gösterilmiştir.



$$V = \int_0^1 \int_x^{2-x} z\,dydx = \int_0^1 \int_x^{2-x} (x^2+y^2)dydx$$

$$= \int_0^1 \left(x^2y + \frac{1}{3}y^3\right)\Big|_x^{2x} dx = \int_0^1 \left(-\frac{8}{3}x^3 + 4x^2 - 4x + \frac{8}{3}\right)dx$$

$$= \left(-\frac{2}{3}x^4 + \frac{4}{3}x^3 - 2x^2 + \frac{8}{3}x\right)\Big|_0^1 = \frac{4}{3} \text{ olur.}$$

5. Aşağıdaki genelleştirilmiş integralleri hesaplayınız.

a) $\int\limits_{1}^{\infty}\int\limits_{e^{-x}}^{1}\frac{1}{x^3y}dydx$ b) $\int\limits_{-1}^{1}\int\limits_{-1/\sqrt{1-x^2}}^{1/\sqrt{1-x^2}}(2y+1)dydx$

c) $\int\limits_{-\infty}^{\infty}\int\limits_{-\infty}^{\infty}\frac{dxdy}{(x^2+1)(y^2+1)}$ d) $\int\limits_{0}^{\infty}\int\limits_{0}^{\infty}xe^{-(x+2y)}dxdy$

Çözüm

a) $I=\int\limits_{1}^{\infty}\int\limits_{e^{-x}}^{1}\frac{1}{x^3y}dydx=\int\limits_{1}^{\infty}\frac{\ln y}{x^3}\Big|_{e^{-x}}^{1}dx=\int\limits_{1}^{\infty}\frac{1}{x^2}dx=\lim\limits_{t\to\infty}\int\limits_{1}^{t}x^{-2}dx=\lim\limits_{t\to\infty}\left(1-\frac{1}{t}\right)=1$

b) $I=\int\limits_{-1}^{1}\int\limits_{-\frac{1}{\sqrt{1-x^2}}}^{\frac{1}{\sqrt{1-x^2}}}(2y+1)dydx=\int\limits_{-1}^{1}(y^2+y)\Bigg|_{\frac{-1}{\sqrt{1-x^2}}}^{\frac{1}{\sqrt{1-x^2}}}=\int\limits_{-1}^{1}\left(\frac{1}{1-x^2}-\frac{1}{1-x^2}+\frac{1}{\sqrt{1-x^2}}+\frac{1}{\sqrt{1-x^2}}\right)dx$

$=\int\limits_{-1}^{1}\frac{2}{\sqrt{1-x^2}}dx=\int\limits_{-1}^{0}\frac{2}{\sqrt{1-x^2}}dx+\int\limits_{0}^{1}\frac{2}{\sqrt{1-x^2}}=\lim\limits_{\varepsilon_1\to 0}\int\limits_{-1+\varepsilon_1}^{0}\frac{2}{\sqrt{1-x^2}}dx+\lim\limits_{\varepsilon_2\to 0}\int\limits_{0}^{1-\varepsilon_2}\frac{2}{\sqrt{1-x^2}}dx$

$=\lim\limits_{\varepsilon_1\to 0}2\arcsin x\,\Big|_{-1+\varepsilon_1}^{0}+\lim\limits_{\varepsilon_2\to 0}2\arcsin x\,\Big|_{0}^{1-\varepsilon_2}$

$=\lim\limits_{\varepsilon_1\to 0}2\arcsin 0-2\arcsin(-1+\varepsilon_1)+\lim\limits_{\varepsilon_2\to 0}2\arcsin(1-\varepsilon_2)-2\arcsin 0$

$=2\arcsin(-1)+2\arcsin 1=-2\frac{3\pi}{2}+2\frac{\pi}{2}=-2\pi$

c) $I=\int\limits_{-\infty}^{\infty}\int\limits_{-\infty}^{\infty}\frac{dxdy}{(x^2+1)(y^2+1)}=\int\limits_{-\infty}^{\infty}\int\limits_{-\infty}^{0}\frac{dxdy}{(x^2+1)(y^2+1)}+\int\limits_{-\infty}^{\infty}\int\limits_{-0}^{\infty}\frac{dxdy}{(x^2+1)(y^2+1)}$

$=\lim\limits_{t\to-\infty}\int\limits_{-\infty}^{\infty}\int\limits_{t}^{0}\frac{dxdy}{(x^2+1)(y^2+1)}+\lim\limits_{s\to\infty}\int\limits_{-\infty}^{\infty}\int\limits_{0}^{s}\frac{dxdy}{(x^2+1)(y^2+1)}$

$=\int\limits_{-\infty}^{\infty}\left(\lim\limits_{t\to\infty}\arctan 0-\arctan t\right)\frac{1}{y^2+1}dy+\int\limits_{-\infty}^{\infty}\lim\limits_{n\to\infty}(\arctan s-\arctan 0)\frac{1}{y^2+1}dy$

$=\frac{\pi}{2}\int\limits_{-\infty}^{\infty}\frac{1}{y^2+1}dy+\frac{\pi}{2}\int\limits_{-\infty}^{\infty}\frac{1}{y^2+1}dy=\pi\int\limits_{-\infty}^{\infty}\frac{1}{y^2+1}dy=\pi\left(\int\limits_{-\infty}^{0}\frac{dy}{y^2+1}+\int\limits_{0}^{\infty}\frac{dy}{y^2+1}\right)$

$=\pi\left(\lim\limits_{m\to-\infty}\int\limits_{m}^{0}\frac{1}{y^2+1}dy+\lim\limits_{n\to\infty}\int\limits_{0}^{n}\frac{1}{y^2+1}du\right)=\pi\left(\frac{\pi}{2}+\frac{\pi}{2}\right)=\pi.\pi=\pi^2$

d) $$I=\int_0^\infty\int_0^\infty xe^{-(x+2y)}dxdy=\int_0^\infty \lim_{t\to\infty}\int_0^t xe^{-(x+2y)}dydx=\int_0^\infty \lim_{t\to\infty}-\frac{x}{2}e^{-(x+2y)}\Big|_t^0 dx$$

$$=\int_0^\infty \lim_{t\to\infty}\left[-\frac{x}{2}e^{-x}-\frac{x}{2}e^{-(x+2t)}\right]dx=\int_0^\infty \frac{x}{2}e^{-x}dx=\lim_{m\to\infty}\frac{1}{2}\int_0^m xe^{-x}dx$$

$$=\lim_{m\to\infty}\frac{1}{2}\left\{-xe^{-x}\Big|_0^m-\int_0^m e^{-x}dx\right\}=\lim_{m\to\infty}\frac{1}{2}\left\{-\frac{t}{e^m}-\frac{1}{e^m}+1\right\}=\frac{1}{2}$$

6. **f, B üzerinde integrallenebilir olsun.**

$$\frac{1}{\text{Alan } B}\iint_B f(x,y)dxdy$$

ifadesine f fonksiyonunun B üzerindeki avarajı (ortalama değeri) denir. Aşağıdaki fonksiyonların yanlarında yazılı bölgeler üzerindeki ortalama değerini bulunuz.

a) $f(x,y)=\sin(x+y)$, $\quad B=\{(x,y):\ 0\le x\le\pi,\ 0\le y\le\pi\}$

b) $f(x,y)=x^2+y^2$, $\quad B=\{(x,y):\ x^2+y^2\le 1\}$

c) $f(x,y)=\dfrac{1}{x^3y}$, $\quad B=\{(x,y):\ e^{-2x}\le y\le e^{-x},\ x\ge 1\}$

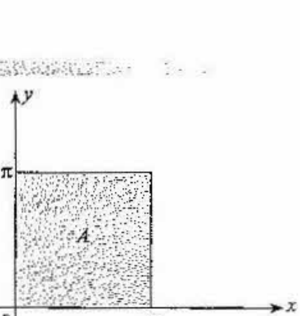
Çözüm

a) $f(x,y)=\sin(x+y)$, $\quad B=\{(x,y):\ 0\le x\le\pi,\ 0\le y\le\pi\}$

$$AlanB=\int_0^\pi\int_0^\pi \sin(x+y)dydx=\pi^2$$

$$\iint_B f(x,y)dxdy=\int_0^\pi\int_0^\pi \sin(x+y)dydx=\int_0^\pi -\cos(x+y)\Big|_0^\pi dx$$

$$=\int_0^\pi[-\cos(x+\pi)+\cos x]dx=-\sin(x+\pi)+\sin x\Big|_0^\pi=0\Rightarrow \text{avaraj}=0$$



b) $f(x,y)=x^2+y^2$ $\quad B=\{(x,y): x^2+y^2\le 1\}$

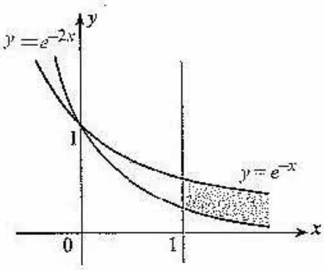
$$AlanB=\int_0^{2\pi}\int_{r=0}^1 rdrd\theta=\pi \text{ ve } \iint_B(x^2+y^2)dxdy=\int_0^{2\pi}\int_0^1 r^2rdrd\varphi=\frac{\pi}{2}$$

olduğundan $Avaraj=\dfrac{\frac{\pi}{2}}{\pi}=\dfrac{1}{2}$ olur.

c) $$\int_1^{\infty}\int_{e^{-2x}}^{e^{-x}} x^{-3}y\,dy\,dx = \frac{1}{2}\int_1^{\infty}(e^{-2x}-e^{-4x})dx$$

$$= \frac{1}{2}\left(-\frac{1}{2}e^{-2x}+\frac{1}{4}e^{-4x}\right)\Big|_1^{\infty} = \frac{1}{2}\left(\frac{e^{-2}}{2}-\frac{1}{4}e^{-4}\right),$$

$$AlanB = \int_1^{\infty}\int_{e^{-2x}}^{e^{-x}} dy\,dx = \int_1^{\infty}(e^{-x}-e^{-2x})dx = e^{-1}-\frac{1}{2}e^{-2},$$



olduğundan avaraj $\dfrac{\frac{1}{4}e^{-2}\left(1-\frac{1}{2}e^{-2}\right)}{e^{-1}\left(1-\frac{1}{2}e^{-1}\right)} = \dfrac{1}{4e}\cdot\dfrac{1-\frac{1}{2}e^{-2}}{1-\frac{1}{2}e^{-1}} = \dfrac{1}{4e^2}\cdot\dfrac{2e^2-1}{2e-1}$ dır.

7. $f(x,y)=u(x)\,v(y)$ ve $\beta=\{(x,y): a\le x\le b,\ c\le y\le d\}$ olsun.

$$\iint_B f(x,y)\,dx\,dy = \left[\int_a^b u(x)\,dx\right]\left[\int_c^d v(y)\,dy\right]$$

olacağını gösteriniz. Bundan yararlanarak

$$\int_0^{\infty} e^{-x^2}\,dx = \frac{\sqrt{\pi}}{2}$$

olduğunu gösteriniz.

Çözüm

$$\iint_B f(x,y)dxdy = \iint_B u(x)v(y)dydx = \int_a^b u(x)\left[\int_c^d v(y)dy\right]dx$$

olur. $\int_c^d v(y)dy$ bir sabit sayı olduğundan integralin dışına çıkarılabilir. Bu durumda

$$\iint_B u(x)v(y)dxdy = \left[\int_c^d v(y)dy\right]\cdot\left[\int_a^b u(x)dx\right]$$ bulunur.

$$I^2 = \left(\int_0^{\infty} e^{-x^2}dx\right)\left(\int_0^{\infty} e^{-y^2}dy\right) = \int_0^{\infty}\int_0^{\infty} e^{-x^2-y^2}dxdy = \int_{\varphi=0}^{\frac{\pi}{2}}\int_{r=0}^{\infty} e^{-r^2}r\,dr\,d\varphi = -\frac{1}{2}\int_0^{\frac{\pi}{2}} e^{-r^2}\Big|_0^{\infty} dr$$

$$= -\frac{1}{2}(-1)\frac{\pi}{2} = \frac{\pi}{4} \Rightarrow I = \frac{\sqrt{\pi}}{2}$$ *bulunur.*

8. Aşağıdaki integralleri hesaplayınız.

a) $\int_0^{\infty} \frac{e^{-x}}{\sqrt{x}} dx$ b) $\int_1^3 \int_0^{\pi/3} \sqrt{1+e^y} \tan x \, dx \, dy$

Çözüm

a) $x = u^2 \Rightarrow dx = 2udu$ olur. Bu durumda

$$\int_0^{\infty} \frac{e^{-x}}{\sqrt{x}} dx = \int_0^{\infty} \frac{e^{-u^2}}{u} \cdot 2udu = 2\int_0^{\infty} e^{-u^2} du = 2\frac{\sqrt{\pi}}{2} = \sqrt{\pi}$$

bulunur.

b) $$I = \int_1^3 \int_0^{\pi/3} \sqrt{1+e^y} \tan x \, dxdy = \left(\int_1^3 \sqrt{1+e^y}\, dy\right) \cdot \left(\int_0^{\pi/3} \tan x \, dx\right)$$ olur.

$$\int_0^{\pi/3} \tan x \, dx = -\ln|\cos x|\Big|_0^{\pi/3} = -\ln\left|\frac{1}{2}\right| - \ln 1 = \ln 2$$

$$\int \sqrt{1+e^y}\, dy = \int t \cdot \frac{2t}{t^2-1} dt = 2\int \left[1 - \frac{1}{t^2-1}\right] dt = 2t + \ln\left|\frac{t+1}{t-1}\right|$$

olduğundan

$$\int_1^3 \sqrt{1+e^y}\, dy = 2(\sqrt{1+e^2} - \sqrt{1+e}) + \ln\left|\frac{\sqrt{1+e^3}+1}{\sqrt{1+e^3}-1}\right|$$ dır.

$$I = \ln 2\left[2(\sqrt{1+e^3} - \sqrt{1+e}) + \ln\left|\frac{\sqrt{1+e^3}+1}{\sqrt{1+e^3-1}}\right|\right]$$ olur.

9. Aşağıdaki integralleri, integrasyon sırasını değiştirerek hesaplayınız.

a) $\int_0^1 \int_0^y (x^2+y^2) dx dy + \int_1^2 \int_0^{2-y} (x^2+y^2)\, dx\, dy$ b) $\int_1^2 \int_x^{x^3} f(x,y) dy dx + \int_1^1 \int_0^{2-y} f(x,y) dx dy$

Çözüm

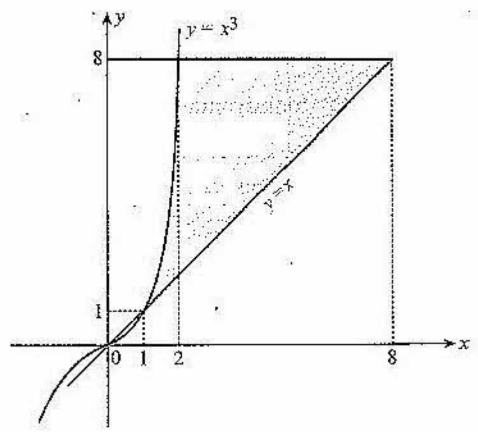
a) 4. sorudan yararlanarak a) şıkkındaki integralin değerinin $\frac{4}{3}$ olduğu kolayca gösterilebilir.

b) İntegrasyon bölgesi sağda renkli olarak gösterilmiştir. Önce x değişkenine göre integral alınırsa bir tek integralle hesap yapılabilir, zira bölge bir yatay basit bölgedir. O halde

$$I = \int_1^8 \int_{\sqrt[3]{y}}^{y} \frac{x^2}{y} dxdy = \int_1^8 \frac{x^3}{3y}\bigg|_{\sqrt[3]{y}}^{y} dy$$

$$= \frac{1}{3}\int_1^8 \left(y^2 - \frac{1}{3}\right)dy = \frac{1}{3}\left(\frac{1}{3}y^3 - \frac{1}{3}y\right)\bigg|_1^8$$

$$= \frac{1}{9}(8^3 - 8 - 1 + 1) = 56$$



10. f fonksiyonu bir D bölgesinde sürekli ve $\forall(x,y) \in D$ için

$$m \le f(x,y) \le M$$

olsun. D bölgesinin alanı A olmak üzere

$$mA \le \iint_D f(x,y)\,dx\,dy \le mA$$

olacağını gösteriniz.

Çözüm

$$m \le f(x,y) \le M \Rightarrow \iint_D m\,dxdy \le \iint_D f(x,y)dxdy \le \iint_D M\,dxdy \Rightarrow$$

$$m\iint_D dxdy \le \iint_D f(x,y)dxdy \le M\iint_D dxdy \Rightarrow mA \le \iint_D f(x,y)dxdy \le mA$$

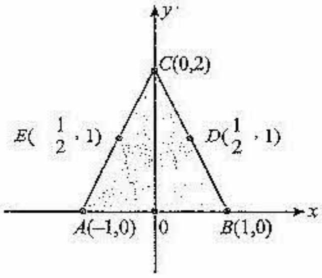
bulunur.

11. Problem 10 dan yararlanak

$$\iint_D e^{x\sin y}\,dx\,dy$$

integrali için bir yaklaşık değer bulunuz. Burada D bölgesi, köşeleri $A(-1, 0)$, $B(1, 0)$, $C(0, 2)$ olan üçgensel bölgedir.

Çözüm



$g(x,y)=e^{x\sin y}$ fonksiyonunun D üzerindeki en küçük ve en büyük değerini bulalım.

$e^{x\sin y}$ ifadesini en büyük ve en küçük yapan x, y değerleri,

$u(x,y)=x\sin y$ ifadesini en büyük ve en küçük yapan x, y değerleridir.

İki değişkenli fonksiyonların kapalı bir bölge üzerindeki ekstremum değerlerinden yararlanarak

$$g_{\min}=g\left(-\frac{1}{2},1\right)=e^{-\frac{1}{2}\sin 1},\quad g_{\max}=g\left(\frac{1}{2},1\right)=e^{\frac{1}{2}\sin 1}$$

olduğu gösterilebilir. Buna göre

$$e^{-\frac{1}{2}\sin 1}\le\iint_D e^{x\sin y}\,dxdy\le e^{\frac{1}{2}\sin 1}$$

olur. Buradaki sayıların yaklaşık değerleri, hesap makineleri yardımıyla hesaplandığında

$$1,10\le\iint_B e^{x\sin y}\,dxdy\le 2,15$$

yazılabilir.

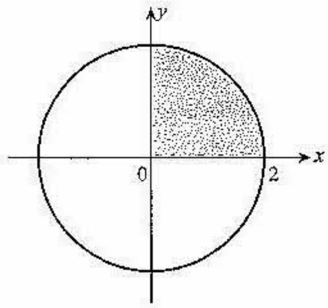
12. **Aşağıdaki integralleri kutupsal koordinatlar yardımıyla hesaplayınız.**

a) $\displaystyle\int_0^2\int_0^{\sqrt{4-x^2}} x\,dy\,dx$ b) $\displaystyle\int_0^2\int_x^{\sqrt{8-x^2}}\frac{dy\,dx}{\sqrt{1+x^2+y^2}}$

c) $\displaystyle\int_0^1\int_0^{\sqrt{1-y^2}} e^{x^2+y^2}\,dx\,dy$ d) $\displaystyle\int_0^2\int_0^{\sqrt{4-y^2}}\frac{dx\,dy}{\sqrt{9-x^2-y^2}}$

Çözüm

a) Verilen integralin integrasyon bölgesi yanda turuncu renkte belirtilmiştir.



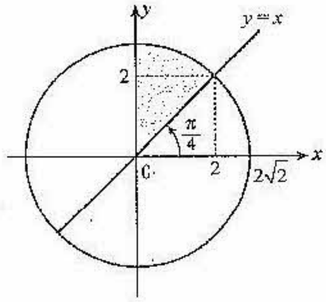
$$I=\int_0^{\frac{\pi}{2}}\int_0^2 r\cos\varphi\, r\,dr\,d\varphi=\int_0^{\frac{\pi}{2}}\frac{r^3}{3}\bigg|_0^2\cos\varphi\,d\varphi=\frac{8}{3}\int_0^{\frac{\pi}{2}}\cos\varphi\,d\varphi$$

$$=\frac{8}{3}\sin\varphi\bigg|_0^{\frac{\pi}{2}}=\frac{8}{3}$$

b) İntegrasyon bölgesi yanda pembe renkte gösterilmiştir.

$$I=\int_{\frac{\pi}{4}}^{\frac{\pi}{2}}\int_{0}^{2\sqrt{2}}\frac{rdrd\varphi}{\sqrt{1+r^2}}=\frac{1}{2}\int_{\frac{\pi}{4}}^{\frac{\pi}{2}}\int_{0}^{2\sqrt{2}}(1+r^2)^{-\frac{1}{2}}2rdrd\varphi$$

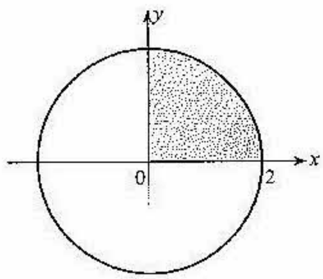
$$=\frac{1}{2}\int_{\frac{\pi}{4}}^{\frac{\pi}{2}}2(1+r^2)^{\frac{1}{2}}\Big|_0^{2\sqrt{2}}d\varphi=\int_{\frac{\pi}{4}}^{\frac{\pi}{2}}2d\varphi=2\varphi\Big|_{\frac{\pi}{4}}^{\frac{\pi}{2}}=\frac{\pi}{2}$$



c) İntegrasyon bölgesi yeşil renkte gösterilmiştir.

$$I=\int_{0}^{\frac{\pi}{2}}\int_{0}^{1}e^{r^2}rdrd\varphi=\frac{1}{2}\int_{0}^{\frac{\pi}{2}}e^{r^2}\Big|_0^1 d\varphi$$

$$=\frac{1}{2}(e-1)\int_{0}^{\frac{\pi}{2}}d\varphi=\frac{\pi}{4}(e-1)$$

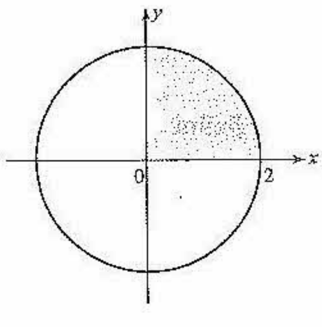


$x=\sqrt{1-y^2}\Rightarrow x^2+y^2=1$

d) İntegrasyon bölgesi yanda turuncu renkte gösterilmiştir.

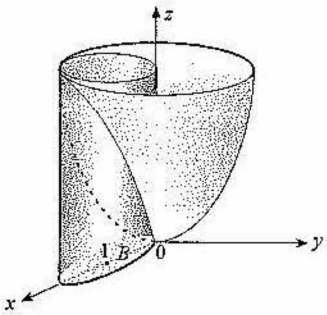
$$I=\int_{0}^{\frac{\pi}{2}}\int_{0}^{2}\frac{rdrd\varphi}{\sqrt{9-r^2}}=\int_{0}^{\frac{\pi}{2}}-\sqrt{9-r^2}\Big|_0^2 d\varphi$$

$$=(-\sqrt{5}+3)\int_{0}^{\frac{\pi}{2}}d\varphi=(3-\sqrt{5})\frac{\pi}{2}$$



13. Üstten $z=x^2+y^2$ paraboloidi, alttan xy düzlemi ve yandan $x^2+y^2=2x$ silindiri tarafından sınırlanan bölgenin hacmini bulunuz.

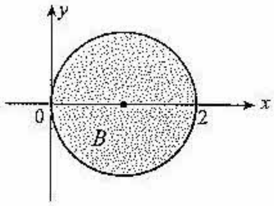
Çözüm



$$V=\iint_B z\,dxdy=\iint_B (x^2+y^2)dxdy$$

$$=\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\int_0^{2\cos\varphi} r^2 r\,dr\,d\varphi=\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{r^4}{4}\Bigg|_0^{2\cos\varphi} d\varphi$$

$$=4\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\cos^4\varphi d\varphi=4\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\cos^2\varphi\cos^2\varphi d\varphi$$



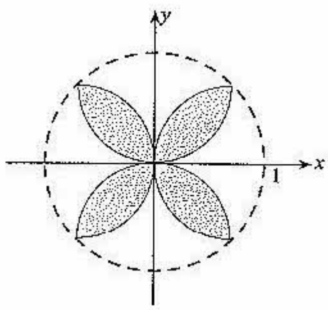
$$=4\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\left(\frac{1+\cos 2\varphi}{2}\right)^2 d\varphi=\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}(1+2\cos 2\varphi+\cos^2 2\varphi)d\varphi$$

$$=\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\left[1+2\cos 2\varphi+\frac{1+\cos 4\varphi}{2}\right]d\varphi=\frac{3}{2}\pi$$

birimküp olur.

14. $r=\sin 2\varphi$ dört yapraklı gülü tarafından sınırlanan bölgenin alanı hesaplayınız.

Çözüm



$$A=\iint_B r\,dr\,d\varphi=4\int_0^{\frac{\pi}{2}}\int_0^{\sin 2\varphi} r\,dr\,d\varphi$$

$$=4\int_0^{\frac{\pi}{2}}\frac{1}{2}r^2\Big|_0^{\sin 2\varphi}d\varphi=2\int_0^{\frac{\pi}{2}}\sin^2 2\varphi d\varphi$$

$$=2\int_0^{\frac{\pi}{2}}\frac{1-\cos 2\varphi}{2}d\varphi=\left(\varphi-\frac{1}{2}\sin 2\varphi\right)\Big|_0^{\frac{\pi}{2}}=\frac{\pi}{2}$$

birimkaredir.

15. B bölgesi $y=\sqrt{4-x^2}$ yarıçemberi ile x ekseni arasında kalan bölge olduğuna göre,

$$\iint\limits_B e^{x^2-y^2}\,dx\,dy$$

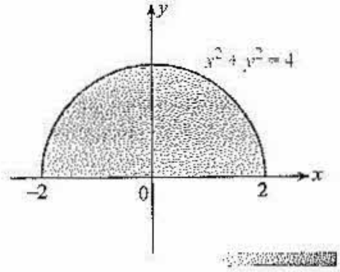
integralini hesaplayınız.

Çözüm

$$I=\iint\limits_B e^{-x^2-y^2}dxdy=\int_0^{\pi}\int_0^{2} e^{-r^2}rdrd\varphi=-\frac{1}{2}\int_0^{\pi}\int_0^{2\pi} e^{-r^2}(-2r)drd\varphi$$

$$=-\frac{1}{2}\int_0^{\pi} e^{-r^2}\Big|_0^2 d\varphi=-\frac{1}{2}(e^{-4}-1)\int_0^{2\pi} d\varphi=(1-e^{-4})\pi$$

olur.



16. Aşağıdaki integralleri hesaplayınız.

a) $\int_0^{\infty} x^2e^{-x^2dx}$ b) $\int_0^{\infty}\sqrt{x}\,e^{-x}\,dx$

Çözüm

a) $I=\int_0^{\infty} x^2e^{-x^2}dx=\int_0^{\infty} y^2e^{-y^2}dy$

$$I^2=\int_0^{\infty}\int_0^{\infty} x^2y^2e^{-x^2-y^2}dxdy=\int_0^{\pi/2}\int_0^{\infty} r^2\sin^2\varphi\cos^2\theta e^{-r^2}rdrd\varphi$$

$$=\left(\int_0^{\pi/2}\sin^2\varphi\cos^2\varphi d\varphi\right)\cdot\left(\int_0^{\infty} r^2e^{-r^2}rdr\right)$$

olur.

$$\int_0^{\infty} r^2e^{-r^2}rdr=\int_0^{\infty} te^{-t}dt=-te^{-t}\Big|_0^{\infty}+\int_0^{\infty} e^{-t}dt \qquad [r^2=t]$$

$$=0+1=1$$

$u=t \quad dv=e^{-t}dt$

$du=dt \quad v=-e^{-t}$

$$\int_0^{\frac{\pi}{2}} \sin^2\varphi\cos^2\varphi\, d\varphi = \int_0^{\frac{\pi}{2}} \left(\frac{1-\cos 2\varphi}{2}\cdot\frac{1+\cos 2\varphi}{2}\right)d\varphi$$

$$= \frac{1}{4}\int_0^{\frac{\pi}{2}} (1-\cos^2 2\varphi)d\varphi = \frac{1}{4}\int_0^{\frac{\pi}{2}} \left(1-\frac{1+\cos 4\varphi}{2}\right)d\varphi$$

$$= \frac{1}{4}\left(\frac{\varphi}{2}-\frac{\sin 4\varphi}{8}\right)\Bigg|_0^{\frac{\pi}{2}} = \frac{\pi}{16}$$

olduğundan

$$I^2 = 1\cdot\frac{\pi}{16} \Rightarrow I = \frac{\sqrt{\pi}}{4}$$

olarak bulunur.

b) $$I = \int_0^{\infty} \sqrt{x}\, e^{-x}dx = \int_0^{\infty} ue^{-u}2udu = 2\int_0^{\infty} u^2e^{-u}du$$

$$= 2\left[-u^2e^{-u}\Big|_0^{\infty} + 2\int_0^{\infty} ue^{-u}du\right] = 2\left[0+2\left(-ue^{-u}\Big|_0^{\infty} + \int_0^{\infty} e^{-u}du\right)\right]$$

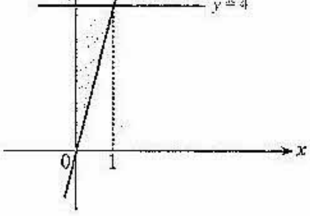
$$= 4\int_0^{\infty} e^{-u}du = 4.1 = 4$$

17. Aşağıdaki integralleri hesaplayınız.

a) $\displaystyle\int_0^1\int_{4x}^4 e^{-y^2}dydx$ b) $\displaystyle\int_0^4\int_{\sqrt{x}}^2 e^{y^3}dydx$

Çözüm

a) İntegrasyon bölgesi yandaki şekilde mavi renkte gösterilmiştir.



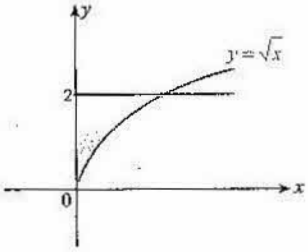
$$\int_0^1\int_{4x}^4 e^{-y^2}dydx = \int_0^4\int_0^{\frac{y}{4}} e^{-y^2}dxdy = \int_0^4 e^{-y^2}x\Big|_0^{\frac{y}{4}}dy$$

$$= \frac{1}{4}\int_0^4 e^{-y^2}ydy = \frac{1}{8}\int_0^{16} e^{-t}dt$$

$$= -\frac{1}{8}e^{-t}\Big|_0^{16} = \frac{1}{8}(1-e^{-16})$$

b) $I = \int\limits_0^2 \int\limits_0^{y^2} e^{y^3} dx dy = \int\limits_0^2 e^{y^3} x \Big|_0^{y^2} dy$

$= \int\limits_0^2 e^{y^3} y^2 dy = \frac{1}{3} e^{y^3} \Big|_0^2 = \frac{1}{3}(e^8 - 1)$



18. B bölgesi $y = \sqrt{x}$ eğrisiyle $x = 0$ ve $y = \sqrt[3]{\pi}$ doğruları tarafından sınırlanan bölge olduğuna göre,

$$I = \iint\limits_B \sin(y^3)\, dx dy$$

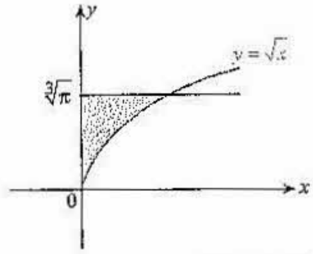
integralini hesaplayınız.

Çözüm

B bölgesi yandaki şekilde yeşil renkte gösterilmiştir.

$I = \int\limits_0^{\sqrt[3]{\pi}} \int\limits_0^{y^2} \sin(y^3) dx dy = \int\limits_0^{\sqrt[3]{\pi}} \sin(y^3) y^2 dy$

$= \frac{1}{3} \int\limits_0^{\pi} \sin t dt = \frac{1}{3}(-\cos t)\Big|_0^{\pi} = \frac{2}{3}$



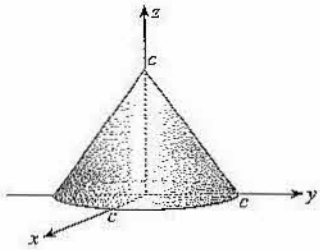
19. Silindirik denklemi $z = c - r$ olan koni ile xy düzlemi arasında kalan bölgenin hacmini hesaplayınız. Burada c pozitif bir sayıdır.

Çözüm

Söz konusu bölge yanda gösterilmiştir.

$V = \int\limits_0^{2\pi} \int\limits_0^{c} (c - r) r dr d\varphi = \int\limits_0^{2\pi} \int\limits_0^{c} (cr - r^2) dr d\varphi = \int\limits_0^{2\pi} \left(c\frac{r^2}{2} - \frac{r^3}{3}\right)\Big|_0^c d\varphi$

$= \left(\frac{c^3}{2} - \frac{c^3}{3}\right) 2\pi = \frac{c^4}{3}\pi$

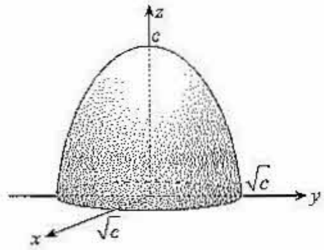


0. $z = c - r^2$ paraboloidi ile xy düzlemi arasında kalan bölgenin hacmini hesaplayınız.

Burada $c > 0$ dır.

Çözüm

öz konusu bölge yanda gösterilmiştir.



$$V = \int_0^{2\pi}\int_0^{\sqrt{c}} (c - r^2) r \, dr \, d\varphi = \frac{\pi}{2} c^2$$

lduğu kolayca gösterilebilir.

1. $z = c - r^n$ yüzeyi ile xy düzlemi arasında kalan bölgenin $V(n, c)$ hacmini bulunuz.

$$\lim_{n \to \infty} V(n,c) = c\pi$$

olacağını gösteriniz.

Çözüm

$$(n,c) = \int_0^{2\pi}\int_0^{\sqrt[n]{c}} (c - r^n) r \, dr \, d\varphi = \int_0^{2\pi} (cr - r^{n+1}) dr \, d\varphi$$

$$= 2\pi\left(c\frac{r^2}{2} - \frac{r^{n+2}}{n+2}\right)\Bigg|_0^{\sqrt[n]{c}} = 2\pi\left(c^{1+\frac{2}{n}} \cdot \frac{n}{2(n+1)}\right)$$

lur.

$\lim_{\to +\infty} V(n,c) = 2\pi\left(c \cdot \frac{1}{2}\right) = c\pi$ bulunur.