CHAPTER 15 MULTIPLE INTEGRALS

15.1 DOUBLE AND ITERATED INTEGRALS OVER RECTANGLES

1. $\int_1^2 \int_0^4 2xy\,dy\,dx = \int_1^2 \left[x\,y^2\right]_0^4 dx = \int_1^2 16x\,dx = \left[8\,x^2\right]_1^2 = 24$

2. $\int_0^2 \int_{-1}^1 (x-y)\,dy\,dx = \int_0^2 \left[xy - \frac{1}{2}y^2\right]_{-1}^{1} dx = \int_0^2 2x\,dx = \left[x^2\right]_0^2 = 4$

3. $\int_{-1}^0 \int_{-1}^1 (x+y+1)\,dx\,dy = \int_{-1}^0 \left[\frac{x^2}{2} + yx + x\right]_{-1}^{1} dy = \int_{-1}^0 (2y+2)\,dy = \left[y^2+2y\right]_{-1}^{0} = 1$

4. $\int_0^1 \int_0^1 \left(1 - \frac{x^2+y^2}{2}\right) dx\,dy = \int_0^1 \left[x - \frac{x^3}{6} - \frac{x\,y^2}{2}\right]_0^1 dy = \int_0^1 \left(\frac{5}{6} - \frac{y^2}{2}\right) dy = \left[\frac{5}{6}y - \frac{y^3}{6}\right]_0^1 = \frac{2}{3}$

5. $\int_0^3 \int_0^2 (4-y^2)\,dy\,dx = \int_0^3 \left[4y - \frac{y^3}{3}\right]_0^2 dx = \int_0^3 \frac{16}{3}\,dx = \left[\frac{16}{3}\,x\right]_0^3 = 16$

6. $\int_0^3 \int_{-2}^0 (x^2y - 2xy)\,dy\,dx = \int_0^3 \left[\frac{x^2y^2}{2} - xy^2\right]_{-2}^{0} dx = \int_0^3 (4x - 2x^2)\,dx = \left[2x^2 - \frac{2x^3}{3}\right]_0^3 = 0$

7. $\int_0^1 \int_0^1 \frac{y}{1+x\,y}\,dx\,dy = \int_0^1 \left[\ln|1+x\,y|\right]_0^1 dy = \int_0^1 \ln|1+y|\,dy = \left[y\ln|1+y| - y + \ln|1+y|\right]_0^1 = 2\ln 2 - 1$

8. $\int_1^4 \int_0^4 \left(\frac{x}{2} + \sqrt{y}\right) dx\,dy = \int_1^4 \left[\frac{1}{4}x^2 + x\sqrt{y}\right]_0^4 dy = \int_1^4 \left(4 + 4\,y^{1/2}\right) dy = \left[4y + \frac{8}{3}y^{3/2}\right]_1^4 = \frac{92}{3}$

9. $\int_0^{\ln 2} \int_1^{\ln 5} e^{2x+y}\,dy\,dx = \int_0^{\ln 2} \left[e^{2x+y}\right]_1^{\ln 5} dx = \int_0^{\ln 2} (5e^{2x} - e^{2x+1})\,dx = \left[\frac{5}{2}e^{2x} - \frac{1}{2}e^{2x+1}\right]_0^{\ln 2} = \frac{3}{2}(5-e)$

10. $\int_0^1 \int_1^2 x\,y\,e^x\,dy\,dx = \int_0^1 \left[\frac{1}{2}x\,y^2 e^x\right]_1^2 dx = \int_0^1 \frac{3}{2}x\,e^x\,dx = \left[\frac{3}{2}x\,e^x - \frac{3}{2}e^x\right]_0^1 = \frac{3}{2}$

11. $\int_{-1}^2 \int_0^{\pi/2} y\sin x\,dx\,dy = \int_{-1}^2 \left[-y\cos x\right]_0^{\pi/2} dy = \int_{-1}^2 y\,dy = \left[\frac{1}{2}y^2\right]_{-1}^2 = \frac{3}{2}$

12. $\int_{\pi}^{2\pi} \int_0^{\pi} (\sin x + \cos y)\,dx\,dy = \int_{\pi}^{2\pi} \left[-\cos x + x\cos y\right]_0^{\pi} dy = \int_{\pi}^{2\pi} (2 + \pi\cos y)\,dy = \left[2y + \pi\sin y\right]_{\pi}^{2\pi} = 2\pi$

13. $\iint_R (6\,y^2 - 2\,x)\,dA = \int_0^1 \int_0^2 (6\,y^2 - 2\,x)\,dy\,dx = \int_0^1 \left[2\,y^3 - 2\,x\,y\right]_0^2 dx = \int_0^1 (16 - 4\,x)\,dx = \left[16\,x - 2\,x^2\right]_0^1 = 14$

14. $\iint_R \frac{\sqrt{x}}{y^2}\,dA = \int_0^4 \int_1^2 \frac{\sqrt{x}}{y^2}\,dy\,dx = \int_0^4 \left[-\frac{\sqrt{x}}{y}\right]_1^2 dx = \int_0^4 \frac{1}{2}x^{1/2}\,dx = \left[\frac{1}{3}x^{3/2}\right]_0^4 = \frac{8}{3}$

15. $\iint_R x\,y\cos y\,dA = \int_{-1}^1 \int_0^{\pi} x\,y\cos y\,dy\,dx = \int_{-1}^1 \left[x\,y\sin y + x\cos y\right]_0^{\pi} dx = \int_{-1}^1 (-2x)\,dx = \left[-x^2\right]_{-1}^1 = 0$

16. $\iint_R y\sin(x+y)\,dA = \int_{-\pi}^0 \int_0^{\pi} y\sin(x+y)\,dy\,dx = \int_{-\pi}^0 \left[-y\cos(x+y) + \sin(x+y)\right]_0^{\pi} dx$
$= \int_{-\pi}^0 (\sin(x+\pi) - \pi\cos(x+\pi) - \sin x)\,dx = \left[-\cos(x+\pi) - \pi\sin(x+\pi) + \cos x\right]_{-\pi}^0 = 4$

17. $\iint_R e^{x-y}\,dA = \int_0^{\ln 2}\int_0^{\ln 2} e^{x-y}\,dy\,dx = \int_0^{\ln 2}\left[-e^{x-y}\right]_0^{\ln 2} dx = \int_0^{\ln 2}\left(-e^{x-\ln 2}+e^x\right)dx = \left[-e^{x-\ln 2}+e^x\right]_0^{\ln 2} = \frac{1}{2}$

18. $\iint_R x\,y\,e^{x y^2}\,dA = \int_0^2\int_0^1 x\,y\,e^{x y^2}\,dy\,dx = \int_0^2\left[\frac{1}{2}e^{x y^2}\right]_0^1 dx = \int_0^2\left(\frac{1}{2}e^x - \frac{1}{2}\right)dx = \left[\frac{1}{2}e^x - \frac{1}{2}x\right]_0^2 = \frac{1}{2}\left(e^2-3\right)$

19. $\iint_R \frac{x\,y^3}{x^2+1}\,dA = \int_0^1\int_0^2 \frac{x\,y^3}{x^2+1}\,dy\,dx = \int_0^1\left[\frac{x\,y^4}{4(x^2+1)}\right]_0^2 dx = \int_0^1 \frac{4x}{x^2+1}\,dx = \left[2\ln|x^2+1|\right]_0^1 = 2\ln 2$

20. $\iint_R \frac{y}{x^2y^2+1}\,dA = \int_0^1\int_0^1 \frac{y}{(x\,y)^2+1}\,dx\,dy = \int_0^1\left[\tan^{-1}(x\,y)\right]_0^1 dy = \int_0^1 \tan^{-1}y\,dy = \left[y\tan^{-1}y - \frac{1}{2}\ln|1+y^2|\right]_0^1 = \frac{\pi}{4} - \frac{1}{2}\ln 2$

21. $\int_1^2\int_1^2 \frac{1}{xy}\,dy\,dx = \int_1^2 \frac{1}{x}(\ln 2 - \ln 1)\,dx = (\ln 2)\int_1^2 \frac{1}{x}\,dx = (\ln 2)^2$

22. $\int_0^1\int_0^{\pi} y\cos xy\,dx\,dy = \int_0^1 \left[\sin xy\right]_0^{\pi} dy = \int_0^1 \sin \pi y\,dy = \left[-\frac{1}{\pi}\cos \pi y\right]_0^1 = -\frac{1}{\pi}(-1-1) = \frac{2}{\pi}$

23. $V = \iint_R f(x,y)\,dA = \int_{-1}^1\int_{-1}^1 (x^2+y^2)\,dy\,dx = \int_{-1}^1\left[x^2 y + \frac{1}{3}y^3\right]_{-1}^1 dx = \int_{-1}^1\left(2x^2+\frac{2}{3}\right)dx = \left[\frac{2}{3}x^3+\frac{2}{3}x\right]_{-1}^1 = \frac{8}{3}$

24. $V = \iint_R f(x,y)\,dA = \int_0^2\int_0^2 (16-x^2-y^2)\,dy\,dx = \int_0^2\left[16y - x^2 y - \frac{1}{3}y^3\right]_0^2 dx = \int_0^2\left(\frac{88}{3}-2x^2\right)dx = \left[\frac{88}{3}x - \frac{2}{3}x^3\right]_0^2$
$= \frac{160}{3}$

25. $V = \iint_R f(x,y)\,dA = \int_0^1\int_0^1 (2-x-y)\,dy\,dx = \int_0^1\left[2y - xy - \frac{1}{2}y^2\right]_0^1 dx = \int_0^1\left(\frac{3}{2}-x\right)dx = \left[\frac{3}{2}x - \frac{1}{2}x^2\right]_0^1 = 1$

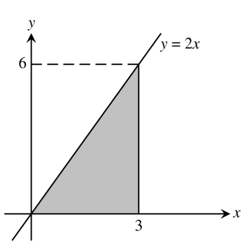
26. $V = \iint_R f(x,y)\,dA = \int_0^4\int_0^2 \frac{y}{2}\,dy\,dx = \int_0^4\left[\frac{y^2}{4}\right]_0^2 dx = \int_0^4 1\,dx = [x]_0^4 = 4$

27. $V = \iint_R f(x,y)\,dA = \int_0^{\pi/2}\int_0^{\pi/4} 2\sin x\cos y\,dy\,dx = \int_0^{\pi/2}\left[2\sin x\sin y\right]_0^{\pi/4} dx = \int_0^{\pi/2}\left(\sqrt{2}\sin x\right)dx = \left[-\sqrt{2}\cos x\right]_0^{\pi/2}$
$= \sqrt{2}$

28. $V = \iint_R f(x,y)\,dA = \int_0^1\int_0^2 (4-y^2)\,dy\,dx = \int_0^1\left[4y - \frac{1}{3}y^3\right]_0^2 dx = \int_0^1\left(\frac{16}{3}\right)dx = \left[\frac{16}{3}x\right]_0^1 = \frac{16}{3}$

15.2 DOUBLE INTEGRALS OVER GENERAL REGIONS

1.



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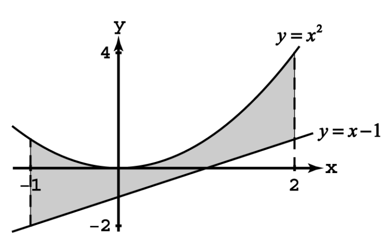

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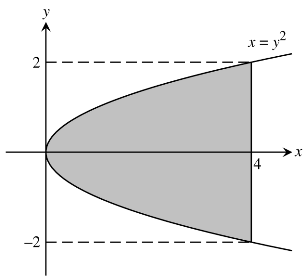

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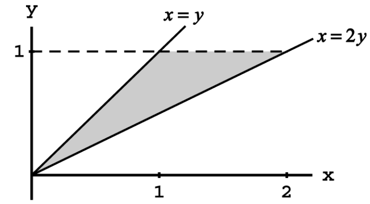

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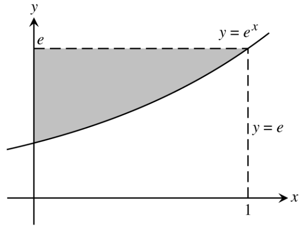

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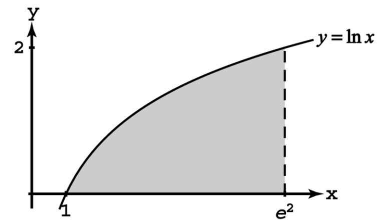

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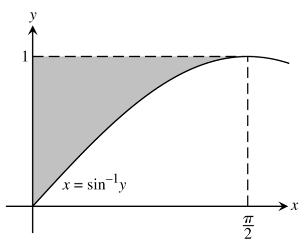

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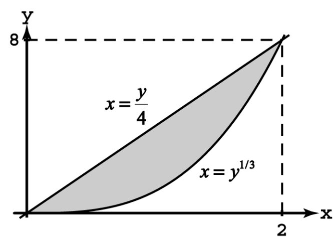


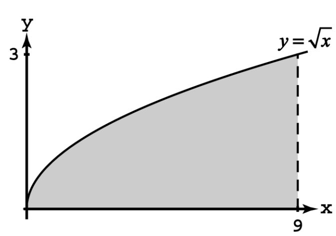
9. (a) $\int_0^2 \int_{x^3}^8 dy\, dx$ (b) $\int_0^8 \int_0^{y^{1/3}} dx\, dy$

10. (a) $\int_0^3 \int_0^{2x} dy\, dx$ (b) $\int_0^6 \int_{y/2}^3 dx\, dy$

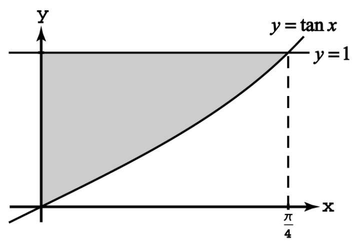
11. (a) $\int_0^3 \int_{x^2}^{3x} dy\, dx$ (b) $\int_0^9 \int_{y/3}^{\sqrt{y}} dx\, dy$

12. (a) $\int_0^2 \int_1^{e^x} dy\, dx$ (b) $\int_1^{e^2} \int_{\ln y}^2 dx\, dy$

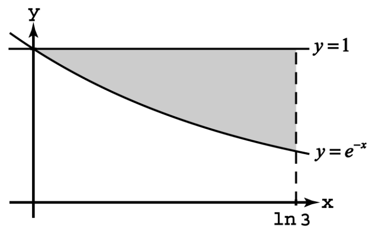
13. (a) $\int_0^9 \int_0^{\sqrt{x}} dy\, dx$
(b) $\int_0^3 \int_{y^2}^9 dx\, dy$



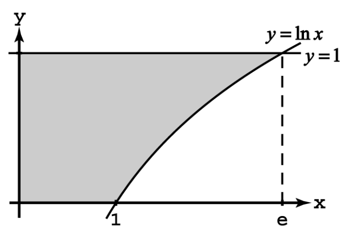
14. (a) $\int_0^{\pi/4} \int_{\tan x}^{1} dy\, dx$
(b) $\int_0^1 \int_0^{\tan^{-1} y} dx\, dy$



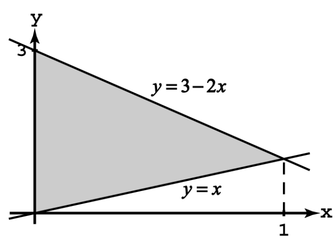
15. (a) $\int_0^{\ln 3} \int_{e^{-x}}^{1} dy\, dx$
(b) $\int_{1/3}^{1} \int_{-\ln y}^{\ln 3} dx\, dy$



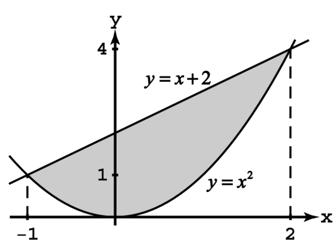
16. (a) $\int_0^1 \int_0^1 dy\, dx + \int_1^e \int_{\ln x}^{1} dy\, dx$
(b) $\int_0^1 \int_0^{e^y} dx\, dy$



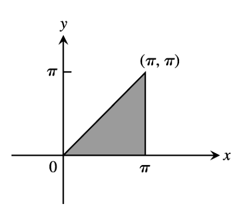
17. (a) $\int_0^1 \int_x^{3-2x} dy\, dx$
(b) $\int_0^1 \int_0^y dx\, dy + \int_1^3 \int_0^{(3-y)/2} dx\, dy$



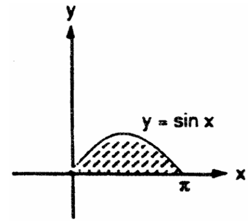
18. (a) $\int_{-1}^{2} \int_{x^2}^{x+2} dy\, dx$
(b) $\int_0^1 \int_{-\sqrt{y}}^{\sqrt{y}} dx\, dy + \int_1^3 \int_{y-2}^{\sqrt{y}} dx\, dy$



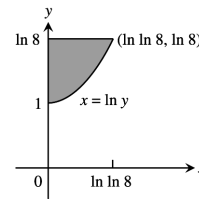
19. $\int_0^{\pi} \int_0^x (x \sin y)\, dy\, dx = \int_0^{\pi} [-x \cos y]_0^x\, dx$
$= \int_0^{\pi} (x - x \cos x)\, dx = \left[\frac{x^2}{2} - (\cos x + x \sin x)\right]_0^{\pi}$
$= \frac{\pi^2}{2} + 2$



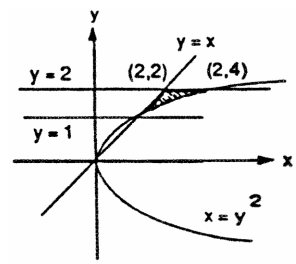
20. $\int_0^{\pi}\int_0^{\sin x} y\,dy\,dx = \int_0^{\pi}\left[\frac{y^2}{2}\right]_0^{\sin x} dx = \int_0^{\pi}\frac{1}{2}\sin^2 x\,dx$
$= \frac{1}{4}\int_0^{\pi}(1-\cos 2x)\,dx = \frac{1}{4}\left[x - \frac{1}{2}\sin 2x\right]_0^{\pi} = \frac{\pi}{4}$



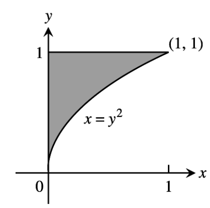
21. $\int_1^{\ln 8}\int_0^{\ln y} e^{x+y}\,dx\,dy = \int_1^{\ln 8}\left[e^{x+y}\right]_0^{\ln y} dy = \int_1^{\ln 8}\left(ye^y - e^y\right) dy$
$= \left[(y-1)e^y - e^y\right]_1^{\ln 8} = 8(\ln 8 - 1) - 8 + e$
$= 8\ln 8 - 16 + e$



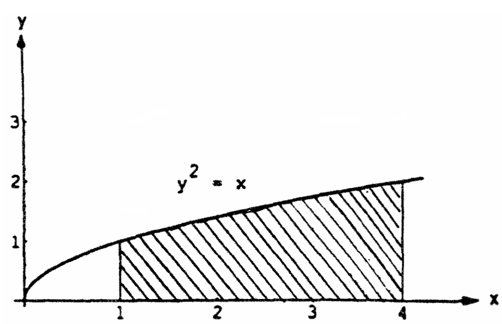
22. $\int_1^2\int_y^{y^2} dx\,dy = \int_1^2 (y^2 - y)\,dy = \left[\frac{y^3}{3} - \frac{y^2}{2}\right]_1^2$
$= \left(\frac{8}{3} - 2\right) - \left(\frac{1}{3} - \frac{1}{2}\right) = \frac{7}{3} - \frac{3}{2} = \frac{5}{6}$



23. $\int_0^1\int_0^{y^2} 3y^3e^{xy}\,dx\,dy = \int_0^1\left[3y^2e^{xy}\right]_0^{y^2} dy$
$= \int_0^1\left(3y^2e^{y^3} - 3y^2\right) dy = \left[e^{y^3} - y^3\right]_0^1 = e - 2$



24. $\int_1^4\int_0^{\sqrt{x}} \frac{3}{2}e^{y/\sqrt{x}}\,dy\,dx = \int_1^4\left[\frac{3}{2}\sqrt{x}\,e^{y/\sqrt{x}}\right]_0^{\sqrt{x}} dx$
$= \frac{3}{2}(e-1)\int_1^4\sqrt{x}\,dx = \left[\frac{3}{2}(e-1)\left(\frac{2}{3}\right)x^{3/2}\right]_1^4 = 7(e-1)$



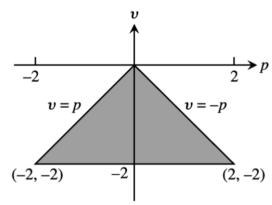
25. $\int_1^2\int_x^{2x} \frac{x}{y}\,dy\,dx = \int_1^2\left[x\ln y\right]_x^{2x} dx = (\ln 2)\int_1^2 x\,dx = \frac{3}{2}\ln 2$

26. $\int_0^1\int_0^{1-x}(x^2+y^2)\,dy\,dx = \int_0^1\left[x^2y + \frac{y^3}{3}\right]_0^{1-x} dx = \int_0^1\left[x^2(1-x) + \frac{(1-x)^3}{3}\right] dx = \int_0^1\left[x^2 - x^3 + \frac{(1-x)^3}{3}\right] dx$
$= \left[\frac{x^3}{3} - \frac{x^4}{4} - \frac{(1-x)^4}{12}\right]_0^1 = \left(\frac{1}{3} - \frac{1}{4} - 0\right) - \left(0 - 0 - \frac{1}{12}\right) = \frac{1}{6}$

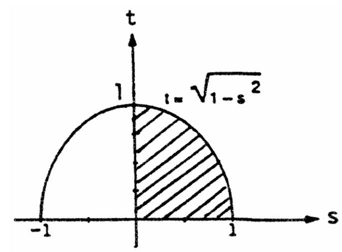
27. $\int_0^1\int_0^{1-u}\left(v - \sqrt{u}\right) dv\,du = \int_0^1\left[\frac{v^2}{2} - v\sqrt{u}\right]_0^{1-u} du = \int_0^1\left[\frac{1-2u+u^2}{2} - \sqrt{u}(1-u)\right] du$
$= \int_0^1\left(\frac{1}{2} - u + \frac{u^2}{2} - u^{1/2} + u^{3/2}\right) du = \left[\frac{u}{2} - \frac{u^2}{2} + \frac{u^3}{6} - \frac{2}{3}u^{3/2} + \frac{2}{5}u^{5/2}\right]_0^1 = \frac{1}{2} - \frac{1}{2} + \frac{1}{6} - \frac{2}{3} + \frac{2}{5} = -\frac{1}{2} + \frac{2}{5} = -\frac{1}{10}$

28. $\int_1^2 \int_0^{\ln t} e^s \ln t \, ds \, dt = \int_1^2 \left[e^s \ln t\right]_0^{\ln t} dt = \int_1^2 (t \ln t - \ln t) \, dt = \left[\frac{t^2}{2} \ln t - \frac{t^2}{4} - t \ln t + t\right]_1^2$
$= (2 \ln 2 - 1 - 2 \ln 2 + 2) - \left(-\frac{1}{4} + 1\right) = \frac{1}{4}$

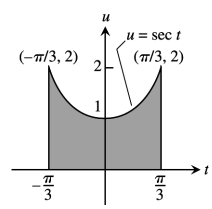
29. $\int_{-2}^0 \int_v^{-v} 2 \, dp \, dv = 2\int_{-2}^0 [p]_v^{-v} \, dv = 2\int_{-2}^0 -2v \, dv$
$= -2\left[v^2\right]_{-2}^0 = 8$



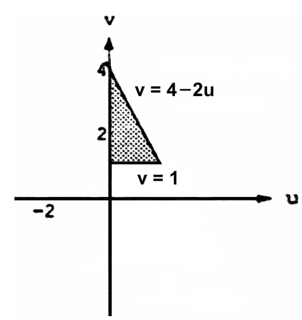
30. $\int_0^1 \int_0^{\sqrt{1-s^2}} 8t \, dt \, ds = \int_0^1 \left[4t^2\right]_0^{\sqrt{1-s^2}} ds$
$= \int_0^1 4\left(1 - s^2\right) ds = 4\left[s - \frac{s^3}{3}\right]_0^1 = \frac{8}{3}$



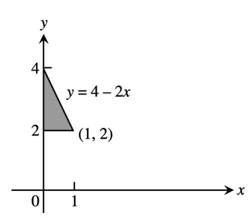
31. $\int_{-\pi/3}^{\pi/3} \int_0^{\sec t} 3 \cos t \, du \, dt = \int_{-\pi/3}^{\pi/3} [(3 \cos t)u]_0^{\sec t}$
$= \int_{-\pi/3}^{\pi/3} 3 \, dt = 2\pi$



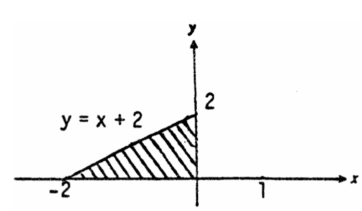
32. $\int_0^{3/2} \int_1^{4-2u} \frac{4-2u}{v^2} \, dv \, du = \int_0^{3/2} \left[\frac{2u-4}{v}\right]_1^{4-2u} du$
$= \int_0^{3/2} (3 - 2u) \, du = \left[3u - u^2\right]_0^{3/2} = \frac{9}{2}$



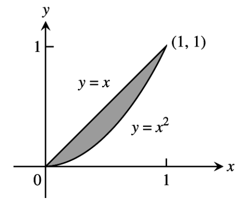
33. $\int_2^4 \int_0^{(4-y)/2} dx \, dy$



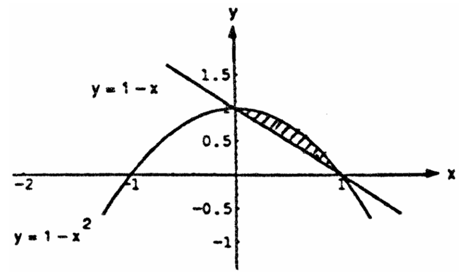
34. $\int_{-2}^0 \int_0^{x+2} dy \, dx$



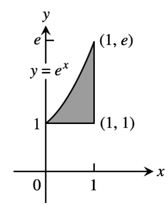
35. $\int_0^1 \int_{x^2}^{x} dy\,dx$



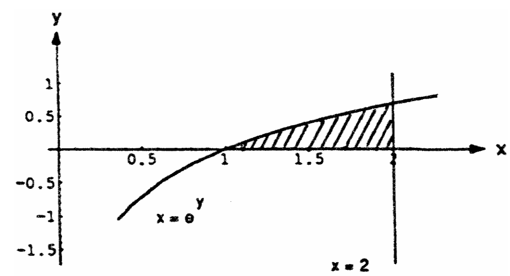
36. $\int_0^1 \int_{1-y}^{\sqrt{1-y}} dx\,dy$



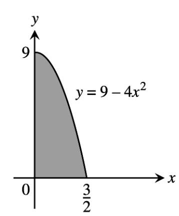
37. $\int_1^e \int_{\ln y}^{1} dx\,dy$



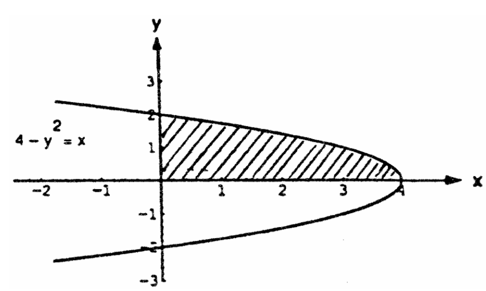
38. $\int_1^2 \int_0^{\ln x} dy\,dx$



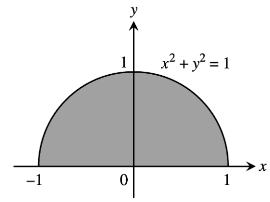
39. $\int_0^9 \int_0^{\frac{1}{2}\sqrt{9-y}} 16x\,dx\,dy$



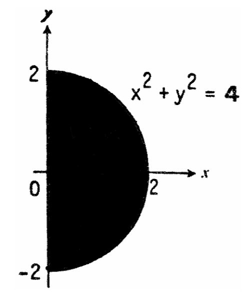
40. $\int_0^4 \int_0^{\sqrt{4-x}} y\,dy\,dx$



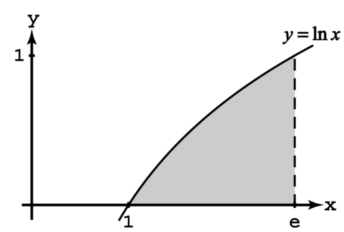
41. $\int_{-1}^{1}\int_{0}^{\sqrt{1-x^2}} 3y\,dy\,dx$



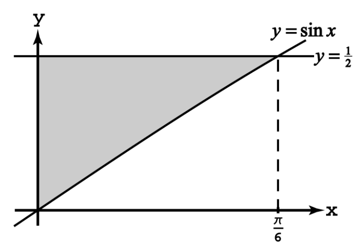
42. $\int_{-2}^{2}\int_{0}^{\sqrt{4-y^2}} 6x\,dx\,dy$



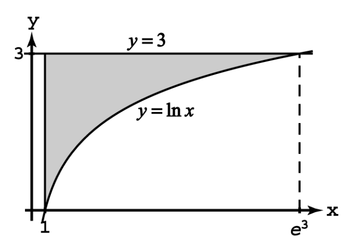
43. $\int_{0}^{1}\int_{e^y}^{e} x\,y\,dx\,dy$



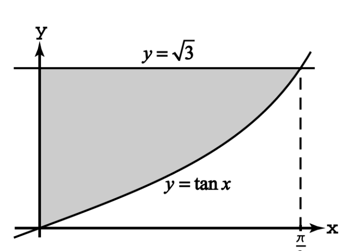
44. $\int_{0}^{1/2}\int_{0}^{\sin^{-1}y} x\,y^2\,dx\,dy$



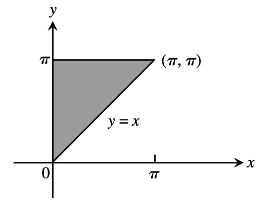
45. $\int_{1}^{e^3}\int_{\ln x}^{3} (x+y)\,dy\,dx$



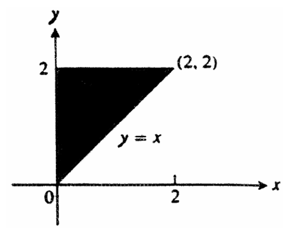
46. $\int_{0}^{\pi/3}\int_{\tan x}^{\sqrt{3}} \sqrt{x\,y}\,dy\,dx$



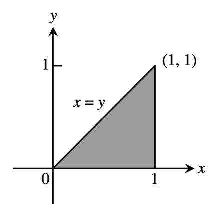
47. $\int_0^{\pi}\int_x^{\pi} \frac{\sin y}{y}\,dy\,dx = \int_0^{\pi}\int_0^{y} \frac{\sin y}{y}\,dx\,dy = \int_0^{\pi} \sin y\,dy = 2$



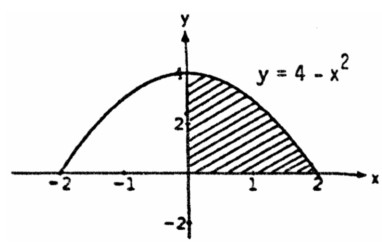
48. $\int_0^2\int_x^2 2y^2 \sin xy\,dy\,dx = \int_0^2\int_0^y 2y^2 \sin xy\,dx\,dy$
$= \int_0^2 [-2y \cos xy]_0^y\,dy = \int_0^2 (-2y \cos y^2 + 2y)\,dy$
$= [-\sin y^2 + y^2]_0^2 = 4 - \sin 4$



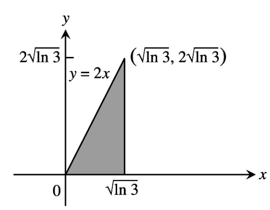
49. $\int_0^1\int_y^1 x^2 e^{xy}\,dx\,dy = \int_0^1\int_0^x x^2 e^{xy}\,dy\,dx = \int_0^1 [xe^{xy}]_0^x\,dx$
$= \int_0^1 (xe^{x^2} - x)\,dx = \left[\frac{1}{2}e^{x^2} - \frac{x^2}{2}\right]_0^1 = \frac{e-2}{2}$



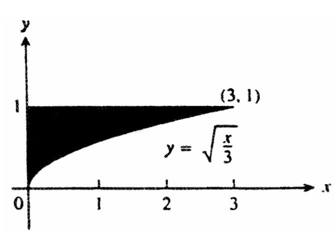
50. $\int_0^2\int_0^{4-x^2} \frac{xe^{2y}}{4-y}\,dy\,dx = \int_0^4\int_0^{\sqrt{4-y}} \frac{xe^{2y}}{4-y}\,dx\,dy$
$= \int_0^4 \left[\frac{x^2e^{2y}}{2(4-y)}\right]_0^{\sqrt{4-y}} dy = \int_0^4 \frac{e^{2y}}{2}\,dy = \left[\frac{e^{2y}}{4}\right]_0^4 = \frac{e^8-1}{4}$



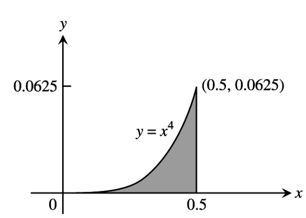
51. $\int_0^{2\sqrt{\ln 3}}\int_{y/2}^{\sqrt{\ln 3}} e^{x^2}\,dx\,dy = \int_0^{\sqrt{\ln 3}}\int_0^{2x} e^{x^2}\,dy\,dx$
$= \int_0^{\sqrt{\ln 3}} 2xe^{x^2}\,dx = [e^{x^2}]_0^{\sqrt{\ln 3}} = e^{\ln 3} - 1 = 2$



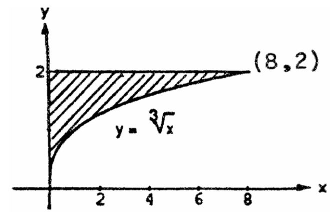
52. $\int_0^3\int_{\sqrt{x/3}}^1 e^{y^3}\,dy\,dx = \int_0^1\int_0^{3y^2} e^{y^3}\,dx\,dy$
$= \int_0^1 3y^2 e^{y^3}\,dy = [e^{y^3}]_0^1 = e - 1$



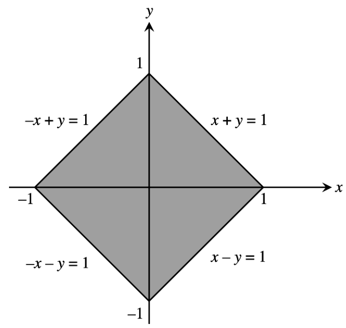
53. $\int_0^{1/16}\int_{y^{1/4}}^{1/2} \cos(16\pi x^5)\,dx\,dy = \int_0^{1/2}\int_0^{x^4} \cos(16\pi x^5)\,dy\,dx$
$= \int_0^{1/2} x^4 \cos(16\pi x^5)\,dx = \left[\frac{\sin(16\pi x^5)}{80\pi}\right]_0^{1/2} = \frac{1}{80\pi}$



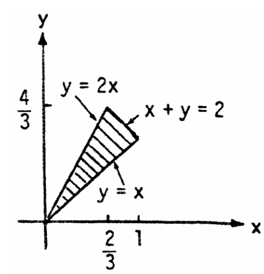
54. $\int_0^8 \int_{\sqrt[3]{x}}^2 \frac{1}{y^4+1}\, dy\, dx = \int_0^2 \int_0^{y^3} \frac{1}{y^4+1}\, dx\, dy$
$= \int_0^2 \frac{y^3}{y^4+1}\, dy = \frac{1}{4}\left[\ln\left(y^4+1\right)\right]_0^2 = \frac{\ln 17}{4}$



55. $\iint_R \left(y - 2x^2\right) dA$
$= \int_{-1}^0 \int_{-x-1}^{x+1} \left(y - 2x^2\right) dy\, dx + \int_0^1 \int_{x-1}^{1-x} \left(y - 2x^2\right) dy\, dx$
$= \int_{-1}^0 \left[\frac{1}{2} y^2 - 2x^2 y\right]_{-x-1}^{x+1} dx + \int_0^1 \left[\frac{1}{2} y^2 - 2x^2 y\right]_{x-1}^{1-x} dx$
$= \int_{-1}^0 \left[\frac{1}{2}(x+1)^2 - 2x^2(x+1) - \frac{1}{2}(-x-1)^2 + 2x^2(-x-1)\right] dx$
$\quad + \int_0^1 \left[\frac{1}{2}(1-x)^2 - 2x^2(1-x) - \frac{1}{2}(x-1)^2 + 2x^2(x-1)\right] dx$
$= -4 \int_{-1}^0 \left(x^3 + x^2\right) dx + 4 \int_0^1 \left(x^3 - x^2\right) dx$
$= -4\left[\frac{x^4}{4} + \frac{x^3}{3}\right]_{-1}^0 + 4\left[\frac{x^4}{4} - \frac{x^3}{3}\right]_0^1 = 4\left[\frac{(-1)^4}{4} + \frac{(-1)^3}{3}\right] + 4\left(\frac{1}{4} - \frac{1}{3}\right) = 8\left(\frac{3}{12} - \frac{4}{12}\right) = -\frac{8}{12} = -\frac{2}{3}$



56. $\iint_R xy\, dA = \int_0^{2/3} \int_x^{2x} xy\, dy\, dx + \int_{2/3}^1 \int_x^{2-x} xy\, dy\, dx$
$= \int_0^{2/3} \left[\frac{1}{2} xy^2\right]_x^{2x} dx + \int_{2/3}^1 \left[\frac{1}{2} xy^2\right]_x^{2-x} dx$
$= \int_0^{2/3} \left(2x^3 - \frac{1}{2} x^3\right) dx + \int_{2/3}^1 \left[\frac{1}{2} x(2-x)^2 - \frac{1}{2} x^3\right] dx$
$= \int_0^{2/3} \frac{3}{2} x^3\, dx + \int_{2/3}^1 \left(2x - x^2\right) dx$



$= \left[\frac{3}{8} x^4\right]_0^{2/3} + \left[x^2 - \frac{2}{3} x^3\right]_{2/3}^1 = \left(\frac{3}{8}\right)\left(\frac{16}{81}\right) + \left(1 - \frac{2}{3}\right) - \left[\frac{4}{9} - \left(\frac{2}{3}\right)\left(\frac{8}{27}\right)\right] = \frac{6}{81} + \frac{27}{81} - \left(\frac{36}{81} - \frac{16}{81}\right) = \frac{13}{81}$

57. $V = \int_0^1 \int_x^{2-x} \left(x^2 + y^2\right) dy\, dx = \int_0^1 \left[x^2 y + \frac{y^3}{3}\right]_x^{2-x} dx = \int_0^1 \left[2x^2 - \frac{7x^3}{3} + \frac{(2-x)^3}{3}\right] dx = \left[\frac{2x^3}{3} - \frac{7x^4}{12} - \frac{(2-x)^4}{12}\right]_0^1$
$= \left(\frac{2}{3} - \frac{7}{12} - \frac{1}{12}\right) - \left(0 - 0 - \frac{16}{12}\right) = \frac{4}{3}$

58. $V = \int_{-2}^1 \int_x^{2-x^2} x^2\, dy\, dx = \int_{-2}^1 \left[x^2 y\right]_x^{2-x^2} dx = \int_{-2}^1 \left(2x^2 - x^4 - x^3\right) dx = \left[\frac{2}{3} x^3 - \frac{1}{5} x^5 - \frac{1}{4} x^4\right]_{-2}^1$
$= \left(\frac{2}{3} - \frac{1}{5} - \frac{1}{4}\right) - \left(-\frac{16}{3} + \frac{32}{5} - \frac{16}{4}\right) = \left(\frac{40}{60} - \frac{12}{60} - \frac{15}{60}\right) - \left(-\frac{320}{60} + \frac{384}{60} - \frac{240}{60}\right) = \frac{189}{60} = \frac{63}{20}$

59. $V = \int_{-4}^1 \int_{3x}^{4-x^2} (x+4)\, dy\, dx = \int_{-4}^1 \left[xy + 4y\right]_{3x}^{4-x^2} dx = \int_{-4}^1 \left[x\left(4 - x^2\right) + 4\left(4 - x^2\right) - 3x^2 - 12x\right] dx$
$= \int_{-4}^1 \left(-x^3 - 7x^2 - 8x + 16\right) dx = \left[-\frac{1}{4} x^4 - \frac{7}{3} x^3 - 4x^2 + 16x\right]_{-4}^1 = \left(-\frac{1}{4} - \frac{7}{3} + 12\right) - \left(\frac{64}{3} - 64\right) = \frac{157}{3} - \frac{1}{4} = \frac{625}{12}$

60. $V = \int_0^2 \int_0^{\sqrt{4-x^2}} (3 - y)\, dy\, dx = \int_0^2 \left[3y - \frac{y^2}{2}\right]_0^{\sqrt{4-x^2}} dx = \int_0^2 \left[3\sqrt{4 - x^2} - \left(\frac{4-x^2}{2}\right)\right] dx$
$= \left[\frac{3}{2} x\sqrt{4 - x^2} + 6 \sin^{-1}\left(\frac{x}{2}\right) - 2x + \frac{x^3}{6}\right]_0^2 = 6\left(\frac{\pi}{2}\right) - 4 + \frac{8}{6} = 3\pi - \frac{16}{6} = \frac{9\pi - 8}{3}$

61. $V = \int_0^2 \int_0^3 \left(4 - y^2\right) dx\, dy = \int_0^2 \left[4x - y^2 x\right]_0^3 dy = \int_0^2 \left(12 - 3y^2\right) dy = \left[12y - y^3\right]_0^2 = 24 - 8 = 16$

62. $V = \int_0^2\int_0^{4-x^2}(4-x^2-y)\,dy\,dx = \int_0^2\left[(4-x^2)\,y-\frac{y^2}{2}\right]_0^{4-x^2}dx = \int_0^2 \frac{1}{2}\,(4-x^2)^2\,dx = \int_0^2\left(8-4x^2+\frac{x^4}{2}\right)dx$
$= \left[8x-\frac{4}{3}\,x^3+\frac{1}{10}\,x^5\right]_0^2 = 16-\frac{32}{3}+\frac{32}{10} = \frac{480-320+96}{30} = \frac{128}{15}$

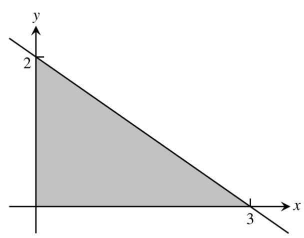
63. $V = \int_0^2\int_0^{2-x}(12-3y^2)\,dy\,dx = \int_0^2[12y-y^3]_0^{2-x}\,dx = \int_0^2[24-12x-(2-x)^3]\,dx = \left[24x-6x^2+\frac{(2-x)^4}{4}\right]_0^2 = 20$

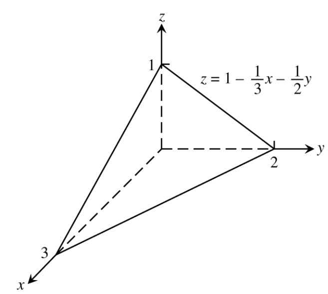
64. $V = \int_{-1}^0\int_{-x-1}^{x+1}(3-3x)\,dy\,dx + \int_0^1\int_{x-1}^{1-x}(3-3x)\,dy\,dx = 6\int_{-1}^0(1-x^2)\,dx + 6\int_0^1(1-x)^2\,dx = 4+2 = 6$

65. $V = \int_1^2\int_{-1/x}^{1/x}(x+1)\,dy\,dx = \int_1^2[xy+y]_{-1/x}^{1/x}\,dx = \int_1^2\left[1+\frac{1}{x}-\left(-1-\frac{1}{x}\right)\right]dx = 2\int_1^2\left(1+\frac{1}{x}\right)dx = 2\,[x+\ln x]_1^2$
$= 2(1+\ln 2)$

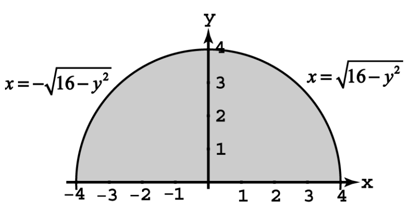
66. $V = 4\int_0^{\pi/3}\int_0^{\sec x}(1+y^2)\,dy\,dx = 4\int_0^{\pi/3}\left[y+\frac{y^3}{3}\right]_0^{\sec x}dx = 4\int_0^{\pi/3}\left(\sec x+\frac{\sec^3 x}{3}\right)dx$
$= \frac{2}{3}\,[7\ln|\sec x+\tan x|+\sec x\tan x]_0^{\pi/3} = \frac{2}{3}\left[7\ln\left(2+\sqrt{3}\right)+2\sqrt{3}\right]$

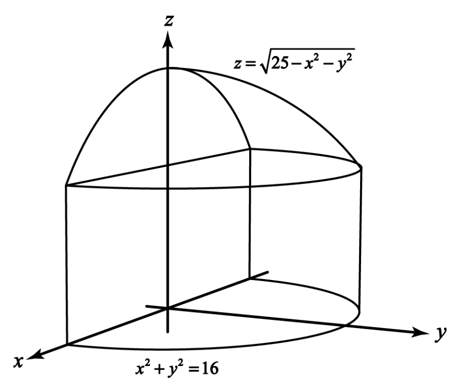
67.





68.





69. $\int_1^\infty\int_{e^{-x}}^1\frac{1}{x^3y}\,dy\,dx = \int_1^\infty\left[\frac{\ln y}{x^3}\right]_{e^{-x}}^1 dx = \int_1^\infty -\left(\frac{-x}{x^3}\right)dx = -\lim_{b\to\infty}\left[\frac{1}{x}\right]_1^b = -\lim_{b\to\infty}\left(\frac{1}{b}-1\right) = 1$

70. $\int_{-1}^1\int_{-1/\sqrt{1-x^2}}^{1/\sqrt{1-x^2}}(2y+1)\,dy\,dx = \int_{-1}^1[y^2+y]\Big|_{-1/(1-x^2)^{1/2}}^{1/(1-x^2)^{1/2}}\,dx = \int_{-1}^1\frac{2}{\sqrt{1-x^2}}\,dx = 4\lim_{b\to 1^-}[\sin^{-1}x]_0^b$
$= 4\lim_{b\to 1^-}[\sin^{-1}b-0] = 2\pi$

71. $\int_{-\infty}^\infty\int_{-\infty}^\infty\frac{1}{(x^2+1)(y^2+1)}\,dx\,dy = 2\int_0^\infty\left(\frac{2}{y^2+1}\right)\left(\lim_{b\to\infty}\tan^{-1}b-\tan^{-1}0\right)dy = 2\pi\lim_{b\to\infty}\int_0^b\frac{1}{y^2+1}\,dy$
$= 2\pi\left(\lim_{b\to\infty}\tan^{-1}b-\tan^{-1}0\right) = (2\pi)\left(\frac{\pi}{2}\right) = \pi^2$

72. $\int_0^\infty \int_0^\infty xe^{-(x+2y)}\,dx\,dy = \int_0^\infty e^{-2y} \lim_{b\to\infty} \left[-xe^{-x} - e^{-x}\right]_0^b dy = \int_0^\infty e^{-2y} \lim_{b\to\infty} \left(-be^{-b} - e^{-b} + 1\right) dy$
$= \int_0^\infty e^{-2y}\,dy = \frac{1}{2} \lim_{b\to\infty} \left(-e^{-2b} + 1\right) = \frac{1}{2}$

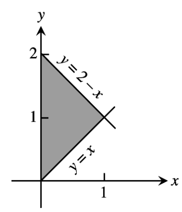
73. $\iint_R f(x,y)\,dA \approx \frac{1}{4}\,f\left(-\frac{1}{2},0\right) + \frac{1}{8}\,f(0,0) + \frac{1}{8}\,f\left(\frac{1}{4},0\right) = \frac{1}{4}\left(-\frac{1}{2}\right) + \frac{1}{8}\left(0 + \frac{1}{4}\right) = -\frac{3}{32}$

74. $\iint_R f(x,y)\,dA \approx \frac{1}{4}\left[f\left(\frac{7}{4},\frac{11}{4}\right) + f\left(\frac{9}{4},\frac{11}{4}\right) + f\left(\frac{7}{4},\frac{13}{4}\right) + f\left(\frac{9}{4},\frac{13}{4}\right)\right] = \frac{1}{16}(29+31+33+35) = \frac{128}{16} = 8$

75. The ray $\theta = \frac{\pi}{6}$ meets the circle $x^2 + y^2 = 4$ at the point $\left(\sqrt{3}, 1\right) \Rightarrow$ the ray is represented by the line $y = \frac{x}{\sqrt{3}}$. Thus,
$\iint_R f(x,y)\,dA = \int_0^{\sqrt{3}} \int_{x/\sqrt{3}}^{\sqrt{4-x^2}} \sqrt{4-x^2}\,dy\,dx = \int_0^{\sqrt{3}} \left[(4-x^2) - \frac{x}{\sqrt{3}}\sqrt{4-x^2}\right] dx = \left[4x - \frac{x^3}{3} + \frac{(4-x^2)^{3/2}}{3\sqrt{3}}\right]_0^{\sqrt{3}} = \frac{20\sqrt{3}}{9}$

76. $\int_2^\infty \int_0^2 \frac{1}{(x^2-x)(y-1)^{2/3}}\,dy\,dx = \int_2^\infty \left[\frac{3(y-1)^{1/3}}{(x^2-x)}\right]_0^2 dx = \int_2^\infty \left(\frac{3}{x^2-x} + \frac{3}{x^2-x}\right) dx = 6\int_2^\infty \frac{dx}{x(x-1)}$
$= 6\lim_{b\to\infty} \int_2^b \left(\frac{1}{x-1} - \frac{1}{x}\right) dx = 6\lim_{b\to\infty} \left[\ln(x-1) - \ln x\right]_2^b = 6\lim_{b\to\infty} \left[\ln(b-1) - \ln b - \ln 1 + \ln 2\right]$
$= 6\left[\lim_{b\to\infty} \ln\left(1 - \frac{1}{b}\right) + \ln 2\right] = 6\ln 2$

77. $V = \int_0^1 \int_x^{2-x} (x^2 + y^2)\,dy\,dx = \int_0^1 \left[x^2 y + \frac{y^3}{3}\right]_x^{2-x} dx$
$= \int_0^1 \left[2x^2 - \frac{7x^3}{3} + \frac{(2-x)^3}{3}\right] dx = \left[\frac{2x^3}{3} - \frac{7x^4}{12} - \frac{(2-x)^4}{12}\right]_0^1$
$= \left(\frac{2}{3} - \frac{7}{12} - \frac{1}{12}\right) - \left(0 - 0 - \frac{16}{12}\right) = \frac{4}{3}$



78. $\int_0^2 (\tan^{-1}\pi x - \tan^{-1} x)\,dx = \int_0^2 \int_x^{\pi x} \frac{1}{1+y^2}\,dy\,dx = \int_0^2 \int_{y/\pi}^{y} \frac{1}{1+y^2}\,dx\,dy + \int_2^{2\pi} \int_{y/\pi}^{2} \frac{1}{1+y^2}\,dx\,dy$
$= \int_0^2 \frac{\left(1-\frac{1}{\pi}\right) y}{1+y^2}\,dy + \int_2^{2\pi} \frac{\left(2-\frac{y}{\pi}\right)}{1+y^2}\,dy = \left(\frac{\pi-1}{2\pi}\right)\left[\ln(1+y^2)\right]_0^2 + \left[2\tan^{-1} y + \frac{1}{2\pi}\ln(1+y^2)\right]_2^{2\pi}$
$= \left(\frac{\pi-1}{2\pi}\right)\ln 5 + 2\tan^{-1} 2\pi - \frac{1}{2\pi}\ln(1+4\pi^2) - 2\tan^{-1} 2 + \frac{1}{2\pi}\ln 5$
$= 2\tan^{-1} 2\pi - 2\tan^{-1} 2 - \frac{1}{2\pi}\ln(1+4\pi^2) + \frac{\ln 5}{2}$

79. To maximize the integral, we want the domain to include all points where the integrand is positive and to exclude all points where the integrand is negative. These criteria are met by the points (x, y) such that $4 - x^2 - 2y^2 \geq 0$ or $x^2 + 2y^2 \leq 4$, which is the ellipse $x^2 + 2y^2 = 4$ together with its interior.

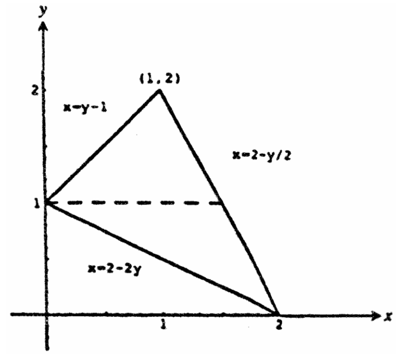
80. To minimize the integral, we want the domain to include all points where the integrand is negative and to exclude all points where the integrand is positive. These criteria are met by the points (x, y) such that $x^2 + y^2 - 9 \leq 0$ or $x^2 + y^2 \leq 9$, which is the closed disk of radius 3 centered at the origin.

81. No, it is not possible. By Fubini's theorem, the two orders of integration must give the same result.

82. One way would be to partition R into two triangles with the line y = 1. The integral of f over R could then be written as a sum of integrals that could be evaluated by integrating first with respect to x and then with respect to y:

$$\iint_R f(x, y)\, dA$$
$$= \int_0^1 \int_{2-2y}^{2-(y/2)} f(x, y)\, dx\, dy + \int_1^2 \int_{y-1}^{2-(y/2)} f(x, y)\, dx\, dy.$$

Partitioning R with the line x = 1 would let us write the integral of f over R as a sum of iterated integrals with order dy dx.



83. $\int_{-b}^{b}\int_{-b}^{b} e^{-x^2-y^2}\, dx\, dy = \int_{-b}^{b}\int_{-b}^{b} e^{-y^2}e^{-x^2}\, dx\, dy = \int_{-b}^{b} e^{-y^2}\left(\int_{-b}^{b} e^{-x^2}\, dx\right) dy = \left(\int_{-b}^{b} e^{-x^2}\, dx\right)\left(\int_{-b}^{b} e^{-y^2}\, dy\right)$

$= \left(\int_{-b}^{b} e^{-x^2}\, dx\right)^2 = \left(2\int_0^b e^{-x^2}\, dx\right)^2 = 4\left(\int_0^b e^{-x^2}\, dx\right)^2$; taking limits as $b \to \infty$ gives the stated result.

84. $\int_0^1\int_0^3 \frac{x^2}{(y-1)^{2/3}}\, dy\, dx = \int_0^3\int_0^1 \frac{x^2}{(y-1)^{2/3}}\, dx\, dy = \int_0^3 \frac{1}{(y-1)^{2/3}}\left[\frac{x^3}{3}\right]_0^1 dy = \frac{1}{3}\int_0^3 \frac{dy}{(y-1)^{2/3}}$

$= \frac{1}{3}\lim_{b \to 1^-}\int_0^b \frac{dy}{(y-1)^{2/3}} + \frac{1}{3}\lim_{b \to 1^+}\int_b^3 \frac{dy}{(y-1)^{2/3}} = \lim_{b \to 1^-}\left[(y-1)^{1/3}\right]_0^b + \lim_{b \to 1^+}\left[(y-1)^{1/3}\right]_b^3$

$= \left[\lim_{b \to 1^-}(b-1)^{1/3} - (-1)^{1/3}\right] - \left[\lim_{b \to 1^+}(b-1)^{1/3} - (2)^{1/3}\right] = (0+1) - \left(0 - \sqrt[3]{2}\right) = 1 + \sqrt[3]{2}$

85-88. Example CAS commands:

Maple:

```
f := (x,y) -> 1/x/y;
q1 := Int( Int( f(x,y), y=1..x ), x=1..3 );
evalf( q1 );
value( q1 );
evalf( value(q1) );
```

89-94. Example CAS commands:

Maple:

```
f := (x,y) -> exp(x^2);
c,d := 0,1;
g1 := y ->2*y;
g2 := y -> 4;
q5 := Int( Int( f(x,y), x=g1(y)..g2(y) ), y=c..d );
value( q5 );
plot3d( 0, x=g1(y)..g2(y), y=c..d, color=pink, style=patchnogrid, axes=boxed, orientation=[-90,0],
        scaling=constrained, title="#89 (Section 15.2)" );
r5 := Int( Int( f(x,y), y=0..x/2 ), x=0..2 ) + Int( Int( f(x,y), y=0..1 ), x=2..4 );
value( r5);
value( q5-r5 );
```

85-94. Example CAS commands:

Mathematica: (functions and bounds will vary)

You can integrate using the built-in integral signs or with the command **Integrate**. In the **Integrate** command, the integration begins with the variable on the right. (In this case, y going from 1 to x).

```
Clear[x, y, f]
f[x_, y_]:= 1 / (x y)
Integrate[f[x, y], {x, 1, 3}, {y, 1, x}]
```

To reverse the order of integration, it is best to first plot the region over which the integration extends. This can be done with ImplicitPlot and all bounds involving both x and y can be plotted. A graphics package must be loaded. Remember to use the double equal sign for the equations of the bounding curves.

```
Clear[x, y, f]
<<Graphics`ImplicitPlot`
ImplicitPlot[{x==2y, x==4, y==0, y==1},{x, 0, 4.1}, {y, 0, 1.1}];
f[x_, y_]:=Exp[x^2]
Integrate[f[x, y], {x, 0, 2}, {y, 0, x/2}] + Integrate[f[x, y], {x, 2, 4}, {y, 0, 1}]
```

To get a numerical value for the result, use the numerical integrator, **NIntegrate**. Verify that this equals the original.

```
Integrate[f[x, y], {x, 0, 2}, {y, 0, x/2}] + NIntegrate[f[x, y], {x, 2, 4}, {y, 0, 1}]
NIntegrate[f[x, y], {y, 0, 1},{x, 2y, 4}]
```

Another way to show a region is with the FilledPlot command. This assumes that functions are given as y = f(x).

```
Clear[x, y, f]
<<Graphics`FilledPlot`
FilledPlot[{x^2, 9},{x, 0,3}, AxesLabels → {x, y}];
f[x_, y_]:= x Cos[y^2]
Integrate[f[x, y], {y, 0, 9}, {x, 0, Sqrt[y]}]
```

85. $\int_1^3 \int_1^x \frac{1}{xy}\, dy\, dx \approx 0.603$

86. $\int_0^1 \int_0^1 e^{-(x^2+y^2)}\, dy\, dx \approx 0.558$

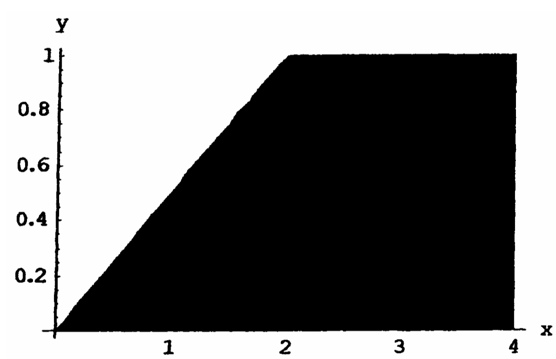
87. $\int_0^1 \int_0^1 \tan^{-1} xy\, dy\, dx \approx 0.233$

88. $\int_{-1}^1 \int_0^{\sqrt{1-x^2}} 3\sqrt{1-x^2-y^2}\, dy\, dx \approx 3.142$

89. Evaluate the integrals:

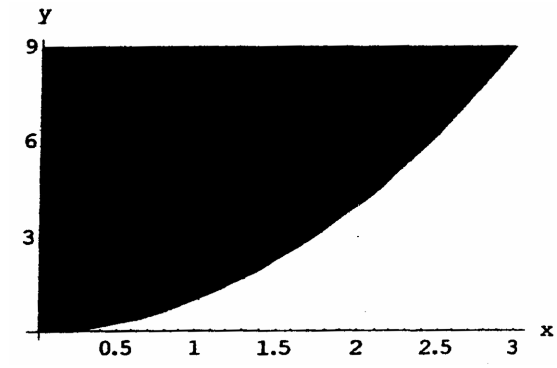
$$\int_0^1 \int_{2y}^4 e^{x^2}\, dx\, dy$$
$$= \int_0^2 \int_0^{x/2} e^{x^2}\, dy\, dx + \int_2^4 \int_0^1 e^{x^2}\, dy\, dx$$
$$= -\tfrac{1}{4} + \tfrac{1}{4}\left(e^4 - 2\sqrt{\pi}\,\mathrm{erfi}(2) + 2\sqrt{\pi}\,\mathrm{erfi}(4)\right)$$
$$\approx 1.1494 \times 10^6$$

The following graph was generated using Mathematica.



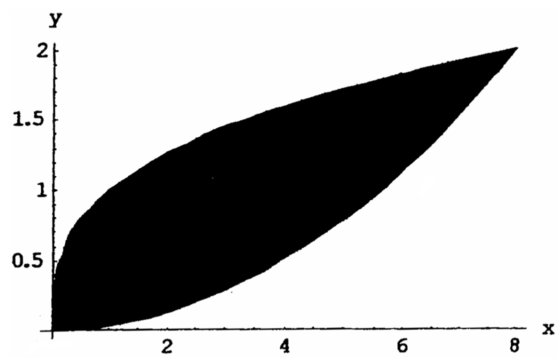
90. Evaluate the integrals:
$\int_0^3 \int_{x^2}^9 x\cos(y^2)\,dy\,dx = \int_0^9 \int_0^{\sqrt{y}} x\cos(y^2)\,dx\,dy$
$= \frac{\sin(81)}{4} \approx -0.157472$

The following graph was generated using Mathematica.



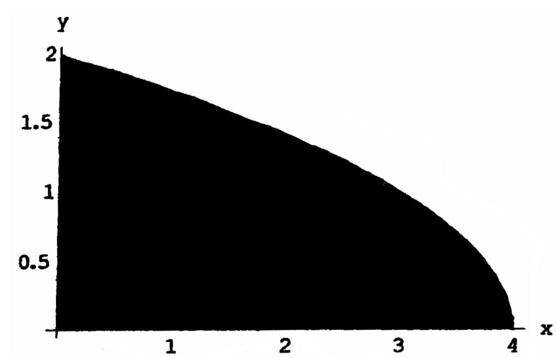
91. Evaluate the integrals:
$\int_0^2 \int_{y^3}^{4\sqrt{2y}} (x^2y - xy^2)\,dx\,dy = \int_0^8 \int_{x^2/32}^{\sqrt[3]{x}} (x^2y - xy^2)\,dy\,dx$
$= \frac{67{,}520}{693} \approx 97.4315$

The following graph was generated using Mathematica.



92. Evaluate the integrals:
$\int_0^2 \int_0^{4-y^2} e^{xy}\,dx\,dy = \int_0^4 \int_0^{\sqrt{4-x}} e^{xy}\,dy\,dx$
≈ 20.5648

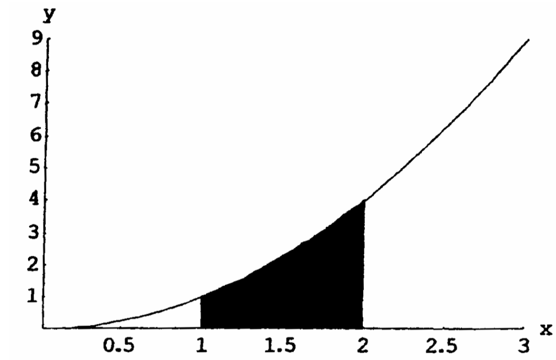
The following graph was generated using Mathematica.



93. Evaluate the integrals:

$$\int_1^2 \int_0^{x^2} \frac{1}{x+y}\, dy\, dx$$
$$= \int_0^1 \int_1^2 \frac{1}{x+y}\, dx\, dy + \int_1^4 \int_{\sqrt{y}}^2 \frac{1}{x+y}\, dx\, dy$$
$$-1 + \ln\left(\frac{27}{4}\right) \approx 0.909543$$

The following graph was generated using Mathematica.



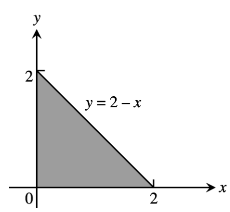
94. Evaluate the integrals:

$$\int_1^2 \int_{y^3}^8 \frac{1}{\sqrt{x^2+y^2}}\, dx\, dy = \int_1^8 \int_1^{\sqrt[3]{x}} \frac{1}{\sqrt{x^2+y^2}}\, dy\, dx$$
$$\approx 0.866649$$

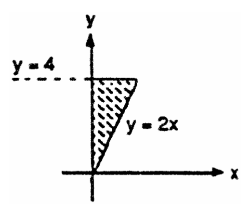
The following graph was generated using Mathematica.

15.3 AREA BY DOUBLE INTEGRATION

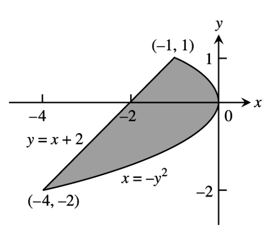
1. $\int_0^2 \int_0^{2-x} dy\, dx = \int_0^2 (2-x)\, dx = \left[2x - \frac{x^2}{2}\right]_0^2 = 2,$
or $\int_0^2 \int_0^{2-y} dx\, dy = \int_0^2 (2-y)\, dy = 2$



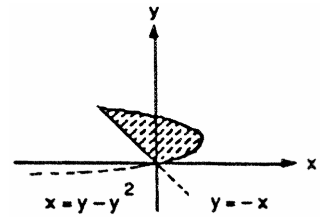
2. $\int_0^2 \int_{2x}^4 dy\, dx = \int_0^2 (4-2x)\, dx = \left[4x - x^2\right]_0^2 = 4,$
or $\int_0^4 \int_0^{y/2} dx\, dy = \int_0^4 \frac{y}{2}\, dy = 4$



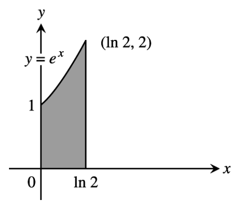
3. $\int_{-2}^1 \int_{y-2}^{-y^2} dx\, dy = \int_{-2}^1 (-y^2 - y + 2)\, dy$
$= \left[-\frac{y^3}{3} - \frac{y^2}{2} + 2y\right]_{-2}^1$
$= \left(-\frac{1}{3} - \frac{1}{2} + 2\right) - \left(\frac{8}{3} - 2 - 4\right) = \frac{9}{2}$



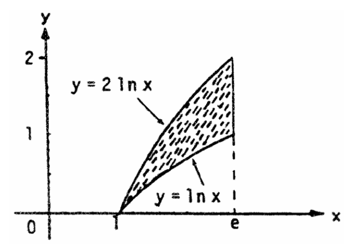
4. $\int_0^2 \int_{-y}^{y-y^2} dx\,dy = \int_0^2 (2y - y^2)\,dy = \left[y^2 - \frac{y^3}{3}\right]_0^2$
$= 4 - \frac{8}{3} = \frac{4}{3}$



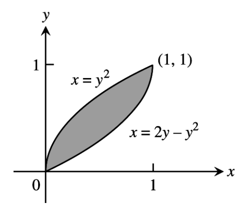
5. $\int_0^{\ln 2} \int_0^{e^x} dy\,dx = \int_0^{\ln 2} e^x\,dx = [e^x]_0^{\ln 2} = 2 - 1 = 1$



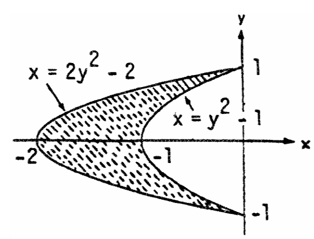
6. $\int_1^e \int_{\ln x}^{2\ln x} dy\,dx = \int_1^e \ln x\,dx = [x \ln x - x]_1^e$
$= (e - e) - (0 - 1) = 1$



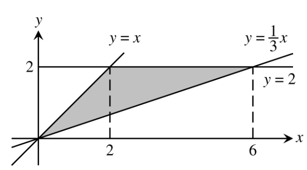
7. $\int_0^1 \int_{y^2}^{2y-y^2} dx\,dy = \int_0^1 (2y - 2y^2)\,dy = \left[y^2 - \frac{2}{3}y^3\right]_0^1$
$= \frac{1}{3}$



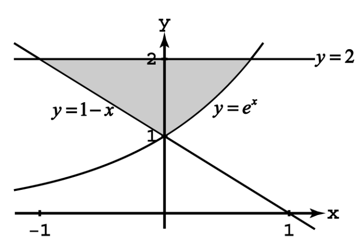
8. $\int_{-1}^1 \int_{2y^2-2}^{y^2-1} dx\,dy = \int_{-1}^1 (y^2 - 1 - 2y^2 + 2)\,dy$
$= \int_{-1}^1 (1 - y^2)\,dy = \left[y - \frac{y^3}{3}\right]_{-1}^1 = \frac{4}{3}$



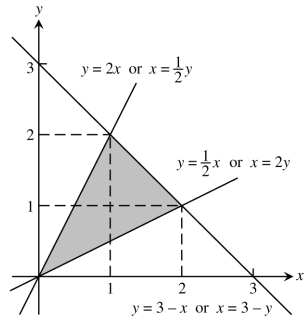
9. $\int_0^2 \int_y^{3y} 1\,dx\,dy = \int_0^2 [x]_y^{3y} dy$
$= \int_0^2 (2y)\,dy = [y^2]_0^2 = 4$



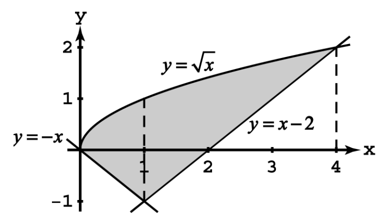
10. $\int_1^2 \int_{1-y}^{\ln y} 1\,dx\,dy = \int_0^2 [x]_{1-y}^{\ln y} dy$
$= \int_1^2 (\ln y - 1 + y)\,dy = \left[y \ln y - 2y + \frac{y^2}{2}\right]_1^2$
$= 2\ln 2 - \frac{1}{2}$



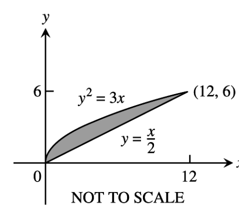
11. $\int_0^1 \int_{x/2}^{2x} 1\,dy\,dx + \int_1^2 \int_{x/2}^{3-x} 1\,dy\,dx$
$= \int_0^1 [y]_{x/2}^{2x} dx + \int_1^2 [y]_{x/2}^{3-x} dx$
$= \int_0^1 \left(\frac{3}{2}x\right) dx + \int_1^2 \left(3 - \frac{3}{2}x\right) dx$
$= \left[\frac{3}{4}x^2\right]_0^1 + \left[3x - \frac{3}{4}x^2\right]_1^2 = \frac{3}{2}$



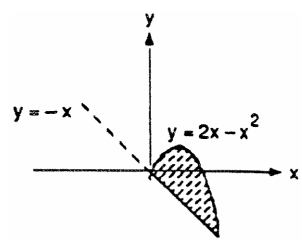
12. $\int_0^1 \int_{-x}^{\sqrt{x}} 1\,dy\,dx + \int_1^4 \int_{x-2}^{\sqrt{x}} 1\,dy\,dx$
$= \int_0^1 [y]_{-x}^{\sqrt{x}} dx + \int_1^4 [y]_{x-2}^{\sqrt{x}} dx$
$= \int_0^1 \left(\sqrt{x} + x\right) dx + \int_1^4 \left(\sqrt{x} - x + 2\right) dx$
$= \left[\frac{2}{3}x^{3/2} + \frac{1}{2}x^2\right]_0^1 + \left[\frac{2}{3}x^{3/2} - \frac{1}{2}x^2 + 2x\right]_1^4 = \frac{13}{3}$



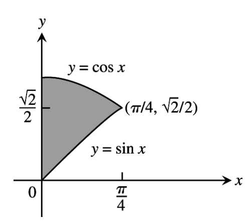
13. $\int_0^6 \int_{y^2/3}^{2y} dx\,dy = \int_0^6 \left(2y - \frac{y^2}{3}\right) dy = \left[y^2 - \frac{y^3}{9}\right]_0^6$
$= 36 - \frac{216}{9} = 12$



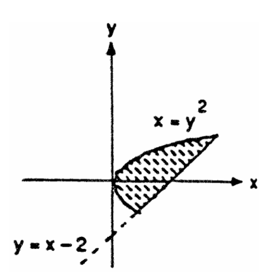
14. $\int_0^3 \int_{-x}^{2x-x^2} dy\,dx = \int_0^3 (3x - x^2)\,dx = \left[\frac{3}{2}x^2 - \frac{1}{3}x^3\right]_0^3$
$= \frac{27}{2} - 9 = \frac{9}{2}$



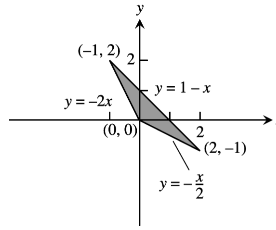
15. $\int_0^{\pi/4} \int_{\sin x}^{\cos x} dy\,dx$
$= \int_0^{\pi/4} (\cos x - \sin x)\,dx = [\sin x + \cos x]_0^{\pi/4}$
$= \left(\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}\right) - (0 + 1) = \sqrt{2} - 1$



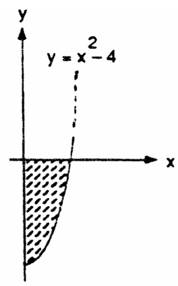
16. $\int_{-1}^2 \int_{y^2}^{y+2} dx\,dy = \int_{-1}^2 (y + 2 - y^2)\,dy = \left[\frac{y^2}{2} + 2y - \frac{y^3}{3}\right]_{-1}^2$
$= \left(2 + 4 - \frac{8}{3}\right) - \left(\frac{1}{2} - 2 + \frac{1}{3}\right) = 5 - \frac{1}{2} = \frac{9}{2}$

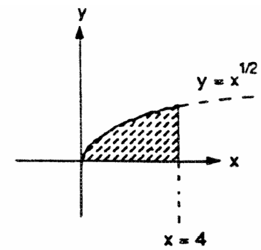


17. $\int_{-1}^{0}\int_{-2x}^{1-x} dy\,dx + \int_{0}^{2}\int_{-x/2}^{1-x} dy\,dx$
$= \int_{-1}^{0}(1+x)\,dx + \int_{0}^{2}\left(1-\frac{x}{2}\right)dx$
$= \left[x+\frac{x^2}{2}\right]_{-1}^{0} + \left[x-\frac{x^2}{4}\right]_{0}^{2} = -\left(-1+\frac{1}{2}\right)+(2-1) = \frac{3}{2}$



18. $\int_{0}^{2}\int_{x^2-4}^{0} dy\,dx + \int_{0}^{4}\int_{0}^{\sqrt{x}} dy\,dx$
$= \int_{0}^{2}(4-x^2)\,dx + \int_{0}^{4} x^{1/2}\,dx$
$= \left[4x-\frac{x^3}{3}\right]_{0}^{2} + \left[\frac{2}{3}x^{3/2}\right]_{0}^{4} = \left(8-\frac{8}{3}\right)+\frac{16}{3} = \frac{32}{3}$





19. (a) average $= \frac{1}{\pi^2}\int_{0}^{\pi}\int_{0}^{\pi}\sin(x+y)\,dy\,dx = \frac{1}{\pi^2}\int_{0}^{\pi}\left[-\cos(x+y)\right]_{0}^{\pi}dx = \frac{1}{\pi^2}\int_{0}^{\pi}\left[-\cos(x+\pi)+\cos x\right]dx$
$= \frac{1}{\pi^2}\left[-\sin(x+\pi)+\sin x\right]_{0}^{\pi} = \frac{1}{\pi^2}\left[(-\sin 2\pi+\sin\pi)-(-\sin\pi+\sin 0)\right] = 0$

(b) average $= \frac{1}{\left(\frac{\pi^2}{2}\right)}\int_{0}^{\pi}\int_{0}^{\pi/2}\sin(x+y)\,dy\,dx = \frac{2}{\pi^2}\int_{0}^{\pi}\left[-\cos(x+y)\right]_{0}^{\pi/2}dx = \frac{2}{\pi^2}\int_{0}^{\pi}\left[-\cos\left(x+\frac{\pi}{2}\right)+\cos x\right]dx$
$= \frac{2}{\pi^2}\left[-\sin\left(x+\frac{\pi}{2}\right)+\sin x\right]_{0}^{\pi} = \frac{2}{\pi^2}\left[\left(-\sin\frac{3\pi}{2}+\sin\pi\right)-\left(-\sin\frac{\pi}{2}+\sin 0\right)\right] = \frac{4}{\pi^2}$

20. average value over the square $= \int_{0}^{1}\int_{0}^{1} xy\,dy\,dx = \int_{0}^{1}\left[\frac{xy^2}{2}\right]_{0}^{1}dx = \int_{0}^{1}\frac{x}{2}\,dx = \frac{1}{4} = 0.25;$

average value over the quarter circle $= \frac{1}{\left(\frac{\pi}{4}\right)}\int_{0}^{1}\int_{0}^{\sqrt{1-x^2}} xy\,dy\,dx = \frac{4}{\pi}\int_{0}^{1}\left[\frac{xy^2}{2}\right]_{0}^{\sqrt{1-x^2}}dx$
$= \frac{2}{\pi}\int_{0}^{1}(x-x^3)\,dx = \frac{2}{\pi}\left[\frac{x^2}{2}-\frac{x^4}{4}\right]_{0}^{1} = \frac{1}{2\pi} \approx 0.159$. The average value over the square is larger.

21. average height $= \frac{1}{4}\int_{0}^{2}\int_{0}^{2}(x^2+y^2)\,dy\,dx = \frac{1}{4}\int_{0}^{2}\left[x^2y+\frac{y^3}{3}\right]_{0}^{2}dx = \frac{1}{4}\int_{0}^{2}\left(2x^2+\frac{8}{3}\right)dx = \frac{1}{2}\left[\frac{x^3}{3}+\frac{4x}{3}\right]_{0}^{2} = \frac{8}{3}$

22. average $= \frac{1}{(\ln 2)^2}\int_{\ln 2}^{2\ln 2}\int_{\ln 2}^{2\ln 2}\frac{1}{xy}\,dy\,dx = \frac{1}{(\ln 2)^2}\int_{\ln 2}^{2\ln 2}\left[\frac{\ln y}{x}\right]_{\ln 2}^{2\ln 2}dx$
$= \frac{1}{(\ln 2)^2}\int_{\ln 2}^{2\ln 2}\frac{1}{x}(\ln 2+\ln\ln 2-\ln\ln 2)\,dx = \left(\frac{1}{\ln 2}\right)\int_{\ln 2}^{2\ln 2}\frac{dx}{x} = \left(\frac{1}{\ln 2}\right)\left[\ln x\right]_{\ln 2}^{2\ln 2}$
$= \left(\frac{1}{\ln 2}\right)(\ln 2+\ln\ln 2-\ln\ln 2) = 1$

23. $\int_{-5}^{5}\int_{-2}^{0}\frac{10{,}000e^y}{1+\frac{|x|}{2}}\,dy\,dx = 10{,}000\left(1-e^{-2}\right)\int_{-5}^{5}\frac{dx}{1+\frac{|x|}{2}} = 10{,}000\left(1-e^{-2}\right)\left[\int_{-5}^{0}\frac{dx}{1-\frac{x}{2}}+\int_{0}^{5}\frac{dx}{1+\frac{x}{2}}\right]$
$= 10{,}000\left(1-e^{-2}\right)\left[-2\ln\left(1-\frac{x}{2}\right)\right]_{-5}^{0} + 10{,}000\left(1-e^{-2}\right)\left[2\ln\left(1+\frac{x}{2}\right)\right]_{0}^{5}$
$= 10{,}000\left(1-e^{-2}\right)\left[2\ln\left(1+\frac{5}{2}\right)\right] + 10{,}000\left(1-e^{-2}\right)\left[2\ln\left(1+\frac{5}{2}\right)\right] = 40{,}000\left(1-e^{-2}\right)\ln\left(\frac{7}{2}\right) \approx 43{,}329$

24. $\int_{0}^{1}\int_{y^2}^{2y-y^2} 100(y+1)\,dx\,dy = \int_{0}^{1}\left[100(y+1)x\right]_{y^2}^{2y-y^2}dy = \int_{0}^{1}100(y+1)\left(2y-2y^2\right)dy = 200\int_{0}^{1}\left(y-y^3\right)dy$
$= 200\left[\frac{y^2}{2}-\frac{y^4}{4}\right]_{0}^{1} = (200)\left(\frac{1}{4}\right) = 50$

25. Let (x_i, y_i) be the location of the weather station in county i for $i = 1, \ldots, 254$. The average temperature in Texas at time t_0 is approximately $\frac{\sum_{i=1}^{254} T(x_i, y_i)\, \Delta_i A}{A}$, where $T(x_i, y_i)$ is the temperature at time t_0 at the weather station in county i, $\Delta_i A$ is the area of county i, and A is the area of Texas.

26. Let $y = f(x)$ be a nonnegative, continuous function on [a, b], then $A = \iint_R dA = \int_a^b \int_0^{f(x)} dy\, dx = \int_a^b [y]_0^{f(x)}\, dx = \int_a^b f(x)\, dx$

15.4 DOUBLE INTEGRALS IN POLAR FORM

1. $x^2 + y^2 = 9^2 \Rightarrow r = 9 \Rightarrow \frac{\pi}{2} \le \theta \le 2\pi,\ 0 \le r \le 9$

2. $x^2 + y^2 = 1^2 \Rightarrow r = 1,\ x^2 + y^2 = 4^2 \Rightarrow r = 4 \Rightarrow -\frac{\pi}{2} \le \theta \le \frac{\pi}{2},\ 1 \le r \le 4$

3. $y = x \Rightarrow \theta = \frac{\pi}{4},\ y = -x \Rightarrow \theta = \frac{3\pi}{4},\ y = 1 \Rightarrow r = \csc\theta \Rightarrow \frac{\pi}{4} \le \theta \le \frac{3\pi}{4},\ 0 \le r \le \csc\theta$

4. $x = 1 \Rightarrow r = \sec\theta,\ y = \sqrt{3}x \Rightarrow \theta = \frac{\pi}{3} \Rightarrow 0 \le \theta \le \frac{\pi}{3},\ 0 \le r \le \sec\theta$

5. $x^2 + y^2 = 1^2 \Rightarrow r = 1,\ x = 2\sqrt{3} \Rightarrow r = 2\sqrt{3}\sec\theta,\ y = 2 \Rightarrow r = 2\csc\theta;\ 2\sqrt{3}\sec\theta = 2\csc\theta \Rightarrow \theta = \frac{\pi}{6}$
$\Rightarrow 0 \le \theta \le \frac{\pi}{6},\ 1 \le r \le 2\sqrt{3}\sec\theta;\ \frac{\pi}{6} \le \theta \le \frac{\pi}{2},\ 1 \le r \le 2\sqrt{3}\csc\theta$

6. $x^2 + y^2 = 2^2 \Rightarrow r = 2,\ x = 1 \Rightarrow r = \sec\theta;\ 2 = \sec\theta \Rightarrow \theta = \frac{\pi}{3}$ or $\theta = -\frac{\pi}{3} \Rightarrow -\frac{\pi}{3} \le \theta \le \frac{\pi}{3},\ \sec\theta \le r \le 2$

7. $x^2 + y^2 = 2x \Rightarrow r = 2\cos\theta \Rightarrow -\frac{\pi}{2} \le \theta \le \frac{\pi}{2},\ 0 \le r \le 2\cos\theta$

8. $x^2 + y^2 = 2y \Rightarrow r = 2\sin\theta \Rightarrow 0 \le \theta \le \pi,\ 0 \le r \le 2\sin\theta$

9. $\int_{-1}^{1}\int_0^{\sqrt{1-x^2}} dy\, dx = \int_0^{\pi}\int_0^1 r\, dr\, d\theta = \frac{1}{2}\int_0^{\pi} d\theta = \frac{\pi}{2}$

10. $\int_0^1 \int_0^{\sqrt{1-y^2}} (x^2 + y^2)\, dx\, dy = \int_0^{\pi/2}\int_0^1 r^3\, dr\, d\theta = \frac{1}{4}\int_0^{\pi/2} d\theta = \frac{\pi}{8}$

11. $\int_0^2 \int_0^{\sqrt{4-y^2}} (x^2 + y^2)\, dx\, dy = \int_0^{\pi/2}\int_0^2 r^3\, dr\, d\theta = 4\int_0^{\pi/2} d\theta = 2\pi$

12. $\int_{-a}^{a}\int_{-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}} dy\, dx = \int_0^{2\pi}\int_0^a r\, dr\, d\theta = \frac{a^2}{2}\int_0^{2\pi} d\theta = \pi a^2$

13. $\int_0^6 \int_0^y x\, dx\, dy = \int_{\pi/4}^{\pi/2}\int_0^{6\csc\theta} r^2\cos\theta\, dr\, d\theta = 72\int_{\pi/4}^{\pi/2} \cot\theta\csc^2\theta\, d\theta = -36\left[\cot^2\theta\right]_{\pi/4}^{\pi/2} = 36$

14. $\int_0^2 \int_0^x y\, dy\, dx = \int_0^{\pi/4}\int_0^{2\sec\theta} r^2\sin\theta\, dr\, d\theta = \frac{8}{3}\int_0^{\pi/4} \tan\theta\sec^2\theta\, d\theta = \frac{4}{3}$

15. $\int_1^{\sqrt{3}}\int_1^x dy\, dx = \int_{\pi/6}^{\pi/4}\int_{\csc\theta}^{\sqrt{3}\sec\theta} r\, dr\, d\theta = \int_{\pi/6}^{\pi/4}\left(\frac{3}{2}\sec^2\theta - \frac{1}{2}\csc^2\theta\right) d\theta = \left[\frac{3}{2}\tan\theta + \frac{1}{2}\cot\theta\right]_{\pi/6}^{\pi/4} = 2 - \sqrt{3}$

16. $\int_{\sqrt{2}}^2\int_{\sqrt{4-y^2}}^{y} dy\, dx = \int_{\pi/4}^{\pi/2}\int_2^{2\csc\theta} r\, dr\, d\theta = \int_{\pi/6}^{\pi/4}(2\csc^2\theta - 2)\, d\theta = \left[-2\cot\theta - \frac{1}{2}\theta\right]_{\pi/4}^{\pi/2} = 2 - \frac{\pi}{2}$

17. $\int_{-1}^{0}\int_{-\sqrt{1-x^2}}^{0} \frac{2}{1+\sqrt{x^2+y^2}}\, dy\, dx = \int_{\pi}^{3\pi/2}\int_0^1 \frac{2r}{1+r}\, dr\, d\theta = 2\int_{\pi}^{3\pi/2}\int_0^1 \left(1-\frac{1}{1+r}\right) dr\, d\theta = 2\int_{\pi}^{3\pi/2} (1-\ln 2)\, d\theta$
$= (1-\ln 2)\pi$

18. $\int_{-1}^{1}\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \frac{2}{(1+x^2+y^2)^2}\, dy\, dx = 4\int_0^{\pi/2}\int_0^1 \frac{2r}{(1+r^2)^2}\, dr\, d\theta = 4\int_0^{\pi/2} \left[-\frac{1}{1+r^2}\right]_0^1 d\theta = 2\int_0^{\pi/2} d\theta = \pi$

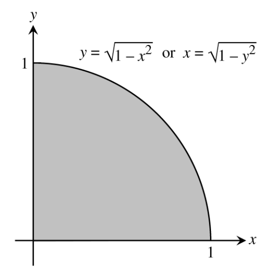
19. $\int_0^{\ln 2}\int_0^{\sqrt{(\ln 2)^2-y^2}} e^{\sqrt{x^2+y^2}}\, dx\, dy = \int_0^{\pi/2}\int_0^{\ln 2} re^r\, dr\, d\theta = \int_0^{\pi/2} (2\ln 2 - 1)\, d\theta = \frac{\pi}{2}(2\ln 2 - 1)$

20. $\int_{-1}^{1}\int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}} \ln(x^2+y^2+1)\, dx\, dy = 4\int_0^{\pi/2}\int_0^1 \ln(r^2+1)\, r\, dr\, d\theta = 2\int_0^{\pi/2} (\ln 4 - 1)\, d\theta = \pi(\ln 4 - 1)$

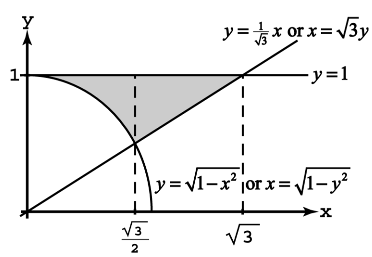
21. $\int_0^1\int_x^{\sqrt{2-x^2}} (x+2y)\, dy\, dx = \int_{\pi/4}^{\pi/2}\int_0^{\sqrt{2}} (r\cos\theta + 2r\sin\theta)\, r\, dr\, d\theta = \int_{\pi/4}^{\pi/2} \left[\frac{r^3}{3}\cos\theta + \frac{2r^3}{3}\sin\theta\right]_0^{\sqrt{2}} d\theta$
$= \int_{\pi/4}^{\pi/2} \left(\frac{2\sqrt{2}}{3}\cos\theta + \frac{4\sqrt{2}}{3}\sin\theta\right) d\theta = \left[\frac{2\sqrt{2}}{3}\sin\theta - \frac{4\sqrt{2}}{3}\cos\theta\right]_{\pi/4}^{\pi/2} = \frac{2(1+\sqrt{2})}{3}$

22. $\int_1^2\int_0^{\sqrt{2x-x^2}} \frac{1}{(x^2+y^2)^2}\, dy\, dx = \int_0^{\pi/4}\int_{\sec\theta}^{2\cos\theta} \frac{1}{r^4}\, r\, dr\, d\theta = \int_0^{\pi/4} \left[-\frac{1}{2r^2}\right]_{\sec\theta}^{2\cos\theta} d\theta = \; = \int_0^{\pi/4} \left(\frac{1}{2}\cos^2\theta - \frac{1}{8}\sec^2\theta\right) d\theta$
$= \left[\frac{1}{4}\theta + \frac{1}{8}\sin 2\theta - \frac{1}{8}\tan\theta\right]_0^{\pi/4} = \frac{\pi}{16}$

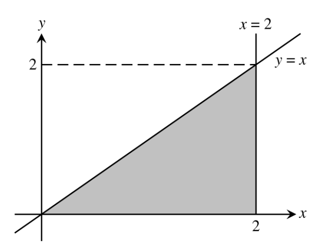
23. $\int_0^1\int_0^{\sqrt{1-x^2}} x\, y\, dy\, dx$ or
$\int_0^1\int_0^{\sqrt{1-y^2}} x\, y\, dx\, dy$



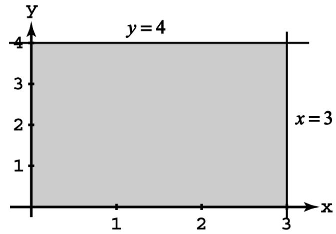
24. $\int_{1/2}^{1}\int_{\sqrt{1-y^2}}^{\sqrt{3}y} x\, dx\, dy$ or
$\int_0^{\sqrt{3}/2}\int_{\sqrt{1-x^2}}^{1} x\, dy\, dx + \int_{\sqrt{3}/2}^{\sqrt{3}}\int_{x/\sqrt{3}}^{1} x\, dy\, dx$



25. $\int_0^2\int_0^x y^2(x^2+y^2)\, dy\, dx$ or
$\int_0^2\int_y^2 y^2(x^2+y^2)\, dx\, dy$



26. $\int_0^3 \int_0^4 \left(x^2 + y^2\right)^3 dy\,dx$ or
$\int_0^4 \int_0^3 \left(x^2 + y^2\right)^3 dx\,dy$



27. $\int_0^{\pi/2} \int_0^{2\sqrt{2-\sin 2\theta}} r\,dr\,d\theta = 2\int_0^{\pi/2} (2 - \sin 2\theta)\,d\theta = 2(\pi - 1)$

28. $A = 2\int_0^{\pi/2} \int_1^{1+\cos\theta} r\,dr\,d\theta = \int_0^{\pi/2} (2\cos\theta + \cos^2\theta)\,d\theta = \frac{8+\pi}{4}$

29. $A = 2\int_0^{\pi/6} \int_0^{12\cos 3\theta} r\,dr\,d\theta = 144\int_0^{\pi/6} \cos^2 3\theta\,d\theta = 12\pi$

30. $A = \int_0^{2\pi} \int_0^{4\theta/3} r\,dr\,d\theta = \frac{8}{9}\int_0^{2\pi} \theta^2\,d\theta = \frac{64\pi^3}{27}$

31. $A = \int_0^{\pi/2} \int_0^{1+\sin\theta} r\,dr\,d\theta = \frac{1}{2}\int_0^{\pi/2} \left(\frac{3}{2} + 2\sin\theta - \frac{\cos 2\theta}{2}\right) d\theta = \frac{3\pi}{8} + 1$

32. $A = 4\int_0^{\pi/2} \int_0^{1-\cos\theta} r\,dr\,d\theta = 2\int_0^{\pi/2} \left(\frac{3}{2} - 2\cos\theta + \frac{\cos 2\theta}{2}\right) d\theta = \frac{3\pi}{2} - 4$

33. average $= \frac{4}{\pi a^2}\int_0^{\pi/2} \int_0^{a} r\sqrt{a^2 - r^2}\,dr\,d\theta = \frac{4}{3\pi a^2}\int_0^{\pi/2} a^3\,d\theta = \frac{2a}{3}$

34. average $= \frac{4}{\pi a^2}\int_0^{\pi/2} \int_0^{a} r^2\,dr\,d\theta = \frac{4}{3\pi a^2}\int_0^{\pi/2} a^3\,d\theta = \frac{2a}{3}$

35. average $= \frac{1}{\pi a^2}\int_{-a}^{a} \int_{-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}} \sqrt{x^2 + y^2}\,dy\,dx = \frac{1}{\pi a^2}\int_0^{2\pi} \int_0^{a} r^2\,dr\,d\theta = \frac{a}{3\pi}\int_0^{2\pi} d\theta = \frac{2a}{3}$

36. average $= \frac{1}{\pi}\iint\limits_R \left[(1 - x)^2 + y^2\right] dy\,dx = \frac{1}{\pi}\int_0^{2\pi} \int_0^{1} \left[(1 - r\cos\theta)^2 + r^2\sin^2\theta\right] r\,dr\,d\theta$
$= \frac{1}{\pi}\int_0^{2\pi} \int_0^{1} (r^3 - 2r^2\cos\theta + r)\,dr\,d\theta = \frac{1}{\pi}\int_0^{2\pi} \left(\frac{3}{4} - \frac{2\cos\theta}{3}\right) d\theta = \frac{1}{\pi}\left[\frac{3}{4}\theta - \frac{2\sin\theta}{3}\right]_0^{2\pi} = \frac{3}{2}$

37. $\int_0^{2\pi} \int_1^{\sqrt{e}} \left(\frac{\ln r^2}{r}\right) r\,dr\,d\theta = \int_0^{2\pi} \int_1^{\sqrt{e}} 2\ln r\,dr\,d\theta = 2\int_0^{2\pi} \left[r\ln r - r\right]_1^{e^{1/2}} d\theta = 2\int_0^{2\pi} \sqrt{e}\left[\left(\frac{1}{2} - 1\right) + 1\right] d\theta = 2\pi\left(2 - \sqrt{e}\right)$

38. $\int_0^{2\pi} \int_1^{e} \left(\frac{\ln r^2}{r}\right) dr\,d\theta = \int_0^{2\pi} \int_1^{e} \left(\frac{2\ln r}{r}\right) dr\,d\theta = \int_0^{2\pi} \left[(\ln r)^2\right]_1^{e} d\theta = \int_0^{2\pi} d\theta = 2\pi$

39. $V = 2\int_0^{\pi/2} \int_1^{1+\cos\theta} r^2\cos\theta\,dr\,d\theta = \frac{2}{3}\int_0^{\pi/2} (3\cos^2\theta + 3\cos^3\theta + \cos^4\theta)\,d\theta$
$= \frac{2}{3}\left[\frac{15\theta}{8} + \sin 2\theta + 3\sin\theta - \sin^3\theta + \frac{\sin 4\theta}{32}\right]_0^{\pi/2} = \frac{4}{3} + \frac{5\pi}{8}$

40. $V = 4\int_0^{\pi/4} \int_0^{\sqrt{2\cos 2\theta}} r\sqrt{2 - r^2}\,dr\,d\theta = -\frac{4}{3}\int_0^{\pi/4} \left[(2 - 2\cos 2\theta)^{3/2} - 2^{3/2}\right] d\theta$
$= \frac{2\pi\sqrt{2}}{3} - \frac{32}{3}\int_0^{\pi/4} (1 - \cos^2\theta)\sin\theta\,d\theta = \frac{2\pi\sqrt{2}}{3} - \frac{32}{3}\left[\frac{\cos^3\theta}{3} - \cos\theta\right]_0^{\pi/4} = \frac{6\pi\sqrt{2} + 40\sqrt{2} - 64}{9}$

41. (a) $I^2 = \int_0^\infty \int_0^\infty e^{-(x^2+y^2)}\,dx\,dy = \int_0^{\pi/2} \int_0^\infty \left(e^{-r^2}\right) r\,dr\,d\theta = \int_0^{\pi/2} \left[\lim\limits_{b\to\infty} \int_0^b re^{-r^2}\,dr\right] d\theta$

$= -\frac{1}{2} \int_0^{\pi/2} \lim\limits_{b\to\infty} \left(e^{-b^2} - 1\right) d\theta = \frac{1}{2} \int_0^{\pi/2} d\theta = \frac{\pi}{4} \Rightarrow I = \frac{\sqrt{\pi}}{2}$

(b) $\lim\limits_{x\to\infty} \int_0^x \frac{2e^{-t^2}}{\sqrt{\pi}}\,dt = \frac{2}{\sqrt{\pi}} \int_0^\infty e^{-t^2}\,dt = \left(\frac{2}{\sqrt{\pi}}\right)\left(\frac{\sqrt{\pi}}{2}\right) = 1$, from part (a)

42. $\int_0^\infty \int_0^\infty \frac{1}{(1+x^2+y^2)^2}\,dx\,dy = \int_0^{\pi/2} \int_0^\infty \frac{r}{(1+r^2)^2}\,dr\,d\theta = \frac{\pi}{2} \lim\limits_{b\to\infty} \int_0^b \frac{r}{(1+r^2)^2}\,dr = \frac{\pi}{4} \lim\limits_{b\to\infty} \left[-\frac{1}{1+r^2}\right]_0^b$

$= \frac{\pi}{4} \lim\limits_{b\to\infty} \left(1 - \frac{1}{1+b^2}\right) = \frac{\pi}{4}$

43. Over the disk $x^2+y^2 \le \frac{3}{4}$: $\iint\limits_R \frac{1}{1-x^2-y^2}\,dA = \int_0^{2\pi} \int_0^{\sqrt{3}/2} \frac{r}{1-r^2}\,dr\,d\theta = \int_0^{2\pi} \left[-\frac{1}{2}\ln(1-r^2)\right]_0^{\sqrt{3}/2} d\theta$

$= \int_0^{2\pi} \left(-\frac{1}{2}\ln\frac{1}{4}\right) d\theta = (\ln 2) \int_0^{2\pi} d\theta = \pi \ln 4$

Over the disk $x^2+y^2 \le 1$: $\iint\limits_R \frac{1}{1-x^2-y^2}\,dA = \int_0^{2\pi} \int_0^1 \frac{r}{1-r^2}\,dr\,d\theta = \int_0^{2\pi} \left[\lim\limits_{a\to 1^-} \int_0^a \frac{r}{1-r^2}\,dr\right] d\theta$

$= \int_0^{2\pi} \lim\limits_{a\to 1^-} \left[-\frac{1}{2}\ln(1-a^2)\right] d\theta = 2\pi \cdot \lim\limits_{a\to 1^-} \left[-\frac{1}{2}\ln(1-a^2)\right] = 2\pi \cdot \infty$, so the integral does not exist over $x^2+y^2 \le 1$

44. The area in polar coordinates is given by $A = \int_\alpha^\beta \int_0^{f(\theta)} r\,dr\,d\theta = \int_\alpha^\beta \left[\frac{r^2}{2}\right]_0^{f(\theta)} d\theta = \frac{1}{2} \int_\alpha^\beta f^2(\theta)\,d\theta = \int_\alpha^\beta \frac{1}{2} r^2\,d\theta$, where $r = f(\theta)$

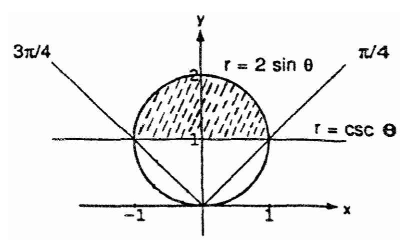
45. average $= \frac{1}{\pi a^2} \int_0^{2\pi} \int_0^a [(r\cos\theta - h)^2 + r^2\sin^2\theta]\,r\,dr\,d\theta = \frac{1}{\pi a^2} \int_0^{2\pi} \int_0^a (r^3 - 2r^2 h\cos\theta + rh^2)\,dr\,d\theta$

$= \frac{1}{\pi a^2} \int_0^{2\pi} \left(\frac{a^4}{4} - \frac{2a^3 h\cos\theta}{3} + \frac{a^2h^2}{2}\right) d\theta = \frac{1}{\pi} \int_0^{2\pi} \left(\frac{a^2}{4} - \frac{2ah\cos\theta}{3} + \frac{h^2}{2}\right) d\theta = \frac{1}{\pi} \left[\frac{a^2\theta}{4} - \frac{2ah\sin\theta}{3} + \frac{h^2\theta}{2}\right]_0^{2\pi}$

$= \frac{1}{2}(a^2 + 2h^2)$

46. $A = \int_{\pi/4}^{3\pi/4} \int_{\csc\theta}^{2\sin\theta} r\,dr\,d\theta = \frac{1}{2} \int_{\pi/4}^{3\pi/4} (4\sin^2\theta - \csc^2\theta)\,d\theta$

$= \frac{1}{2}[2\theta - \sin 2\theta + \cot\theta]_{\pi/4}^{3\pi/4} = \frac{\pi}{2}$



47-50. Example CAS commands:

Maple:

```
f := (x,y) -> y/(x^2+y^2);
    a,b := 0,1;
    f1 := x -> x;
    f2 := x -> 1;
    plot3d( f(x,y), y=f1(x)..f2(x), x=a..b, axes=boxed, style=patchnogrid, shading=zhue, orientation=[0,180], title="#47(a)
        (Section 15.4)" );                                            # (a)
    q1 := eval( x=a, [x=r*cos(theta),y=r*sin(theta)] );              # (b)
    q2 := eval( x=b, [x=r*cos(theta),y=r*sin(theta)] );
    q3 := eval( y=f1(x), [x=r*cos(theta),y=r*sin(theta)] );
    q4 := eval( y=f2(x), [x=r*cos(theta),y=r*sin(theta)] );
    theta1 := solve( q3, theta );
```